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Riemann Surfaces and Algebraic Curves

WiSe 2020/21

Exercise Sheet 1

These exercises will be discussed on November 4

Exercise 1.1 (The ring of germs of holomorphic functions) We consider the ring of complex power series which converge in a neighborhood of zero:

$$\mathbb{C}\{z\} = \left\{ f(z) = \sum_{n\geq 0} a_n z^n \;\middle|\; a_n \in \mathbb{C},\, f(z) \text{ converges in a neighborhood of } 0 \right\}.$$

This models germs of holomorphic functions around zero.

a) Show that an element $f(z) \in \mathbb{C}\{z\}$ is invertible if and only if $f(0) \neq 0$.Conclude that any element can be written in the form

$$f(z) = z^e \cdot g(z), \qquad \text{with } e \geq 0 \text{ and } g(z) \text{ invertible.}$$

In algebraic language, this means that $\mathbb{C}\{z\}$ is a discrete valuation ring, with parameter z.

b) Let $f \in \mathbb{C}\{z\}$ be such that $f(0) = 0$ and $f'(0) \neq 0$. Show that f has a functional inverse, meaning that there exists $g \in \mathbb{C}\{z\}$ such that:

$$(f \circ g)(z) = z, \qquad (g \circ f)(z) = z.$$

In other words, $w = f(z)$ is a local coordinate around zero.

c) Let $f(z) \in \mathbb{C}\{z\}$ be such that $f(0) \neq 0$ and let $e \geq 1$ be an integer. Show that there exists $g(z) \in \mathbb{C}\{z\}$ such that $f(z) = g(z)^e$.

Exercise 1.2 (Local form of holomorphic functions) Let $U \subseteq \mathbb{C}$ be an open neighborhood of zero and $f\colon U \to \mathbb{C}$ a non-constant holomorphic function such that $f(0) = 0$. Show that there is a local coordinate z around 0 such that f can be written as

$$f(z) = z^e, \qquad \text{for a certain integer } e \geq 1.$$

Exercise 1.3 (Holomorphic functions are open) Let $U \subseteq \mathbb{C}$ be open and connected and let $f\colon U \to \mathbb{C}$ be a non-constant holomorphic function. Show that f is open, meaning that images of open sets are again open.