Simple mathematical models for urban growth

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A simple deterministic mathematical model for urban growth and spread is developed in which there is local logistic growth, diffusion, and an integral ('killing') term representing the inhibitory effect of congested central areas on growth. Migration is represented by boundary conditions. The resulting integrodifferential equation has steady-state solutions which satisfy a third-order nonlinear differential equation and are obtained numerically. Time-dependent solutions are obtained by using a modified Crank–Nicolson method. The results indicate that there is a threshold effect with respect to the initial population distribution and that for some parameter values populations do not persist, regardless of the initial numbers. Steady state densities may take an exponentially decaying form as in Clark's original analysis or there may be a maximum some distance from the city centre.

1. Introduction

Patterns of the distribution of populations around city centres are extremely variable. Clearly city development depends on constraints imposed by features of the local geography such as lakes, coastlines and mountain ranges. Rivers have played an important role as centres of attraction and indeed one of the first known cities, Babylon, was located at the junction of the Tigris and Euphrates rivers. The sizes of population centres also exhibit great variability, with numbers from just a few to ten or more million.

Despite these various sizes and shapes, it is somewhat remarkable that Clark (1951) was able to extract a fairly general quantitative pattern of urban population density. Clark's discovery was that population densities, as measured by numbers of individuals per unit area, decline approximately exponentially with distance from city centres, although no strict definition is given of city centre or how to find it. Thus the population density ρ is given at distance r by

$$\rho(r) = A\,\mathrm{e}^{-br}, \quad r \geqslant 0, \tag{1}$$

where A and b are positive constants. It is noteworthy that, according to Montroll & Badger (1974), the exponential density had been used to describe some urban populations in Germany by Bleicher as early as 1892. Clark (1951), however, considered its application to 20 cities and included in his analysis fits for some cities such as London and Paris at multiple time points.

The changes in time of the spatial distribution of population show that a satisfactory mathematical model for urban growth must in fact be a dynamical one.

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Table 1. *Parameters for London*

Year	A	b
1801	269	1.26
1841	279	0.94
1871	224	0.61
1901	170	0.37
1921	115	0.27
1931	123	0.28
1939	83	0.22
1951	62	0.20

The values of A and b reported for London (see Montroll & Badger 1974) are given in table 1, where A is in thousand and b is in miles^{-1}.

These data show that, in general, (i) A decreased, meaning that the central density decreased, and (ii) b decreased, meaning that the population became more decentralized. However, the total population, which is given by

$$N_{\text{tot}} = 2\pi A/b^2, \tag{2}$$

increased, where it is assumed that the city is circularly symmetric.

The usefulness of simple models for urban growth is manifest. For example, if the density (1) is valid, one only needs measurements of local density at two places to predict not only the density at any point, but also the total population or the population is any given region. A dynamical model has the additional power of predicting the future distribution and seeing back to past distributions. The latter could be invaluable in historical and archaeological investigations. On the other hand a satisfactory model for urban growth would be of great assistance in the planning of future cities, because the underlying dynamical processes could be understood and perhaps controlled.

In nearly every case examined, the exponential density did not fit the data well at the centre of the city (Clark 1951; Montroll & Badger 1974). This is attributed to the special nature of the central region as a central business district and leads naturally to a consideration of the daily movements of the population which some authors have incorporated into their models. Thus Mills (1970, 1972) considered not only the residential density but also the employment density, and Ishikawa (1980) considered a model in which there was diffusion of both a daytime and residential population. Other authors have modified Clark's density function (1) to account for the different shapes encountered. Thus Newling (1969) put

$$\rho(r) = A\,\mathrm{e}^{br-cr^2}, \tag{3}$$

where b, $c > 0$, so that there is a maximum density at $r = b/2c$.

The dynamical evolution of cities is very complex so it is not surprising that many different mathematical models have been put forward. Most models have concentrated on economic factors such as transportation costs and housing costs (see, for example, Mills 1970; To *et al.* 1983), but others have concentrated on geometrical factors related to transportation (Krakover & Casetti 1988). A reaction–diffusion model based on predator–prey equations, with rent as the predator and residential density as the prey was proposed by Zhang (1988). There are many known attracting and repelling features of urban developments most of which were listed by Montroll & Badger (1974). A cogent survey of the literature may be found in Thrall (1988).

2. A mathematical model for urban growth and spread

In formulating a deterministic mathematical model for the evolution of a city we have endeavoured to capture the essential features as simply as possible, keeping the number of parameters small. We consider that there are four principal factors which must be incorporated into a model:

(1) local growth;
(2) spread into neighbouring regions;
(3) immigration and emigration; and
(4) diminution of growth caused by increasing congestion of central areas.

To mathematically represent the first of these factors, we choose the classical logistic law (Verhulst 1838). According to this, a local density $\rho(t)$ grows according to the first-order differential equation

$$\mathrm{d}\rho/\mathrm{d}t = k\rho(\sigma - \rho), \tag{4}$$

where k is an intrinsic growth rate and σ is the carrying capacity: i.e. the maximum (equilibrium) attainable population density.

The second factor, spread of the population, is most simply taken into account by diffusion. It is assumed that the region of interest is an ideal one and does not offer impediments to spread: such as is the case of a city growing on a plain. Let the population density (per unit area) at time t be $\rho(\boldsymbol{x}, t)$, where $\boldsymbol{x} = (x_1, x_2)$ is the position vector relative to the city centre. That is, in a region A, the total number of residents at t is given by

$$N(A, t) = \int_A\int \rho(\boldsymbol{x}, t)\, \mathrm{d}S,$$

where $\mathrm{d}S$ is an elemental area. Alternatively one may work in polar coordinates with $\rho = \rho(r, \theta, t)$. Then in the case of circular symmetry, whereby $\rho = \rho(r, t)$ depends only on distance from the centre and we have the same notation as Clark for the number of residents at a distance less than r,

$$N(r, t) = \int_0^r 2\pi r' \rho(r', t)\, \mathrm{d}r'. \tag{5}$$

The model including diffusion and local growth takes the form of a reaction–diffusion equation:

$$\rho_t = D\nabla^2\rho + k\rho(\sigma - \rho), \tag{6}$$

where D is the diffusion coefficient, assumed constant, and the subscript t refers to partial differentiation with respect to t. Equation (6) is a version of that used by Fisher (1937) to describe the spread of an advantageous gene. Explicitly, we have in cartesian coordinates

$$\rho_t = D(\rho_{xx} + \rho_{yy}) + k\rho(\sigma - \rho) \tag{7A}$$

and in polar coordinates,

$$\rho_t = D(\rho_{rr} + r^{-1}\rho_r + r^{-2}\rho_{\theta\theta}) + k\rho(\sigma - \rho). \tag{7B}$$

Thirdly, we incorporate the negative effect that a high density and hence usually congested inner core has on the settlement of residents. This effect is cumulative. We suppose in fact that at a distance r from the city centre it is proportional to the number of individuals who live closer to the centre than r. This represents heuristically the fact that, at a given location, the problems of travel and of finding reasonably priced housing will be greater the larger the population who live between that location and the city centre. This is a simplification but leads to a tractable model without the introduction of explicit variables to represent housing and travel costs. This discouragement or inhibition factor is also assumed to be proportional to the local density, this assumption being rendered plausible by the usual collision argument such as that used in predator–prey modelling with the Lotka–Volterra equations (see, for example, Boyce & DiPrima 1977).

Let us assume circular symmetry so that equation (5) gives the number of residents at distances between 0 and r from the city centre. Then a model equation for the evolution of an urban population, which incorporates this third effect as well as diffusion and logistic growth, is

$$\rho_t = D(\rho_{rr}+r^{-1}\rho_r)+k\rho(\sigma-\rho)-\beta\rho\int_0^r 2\pi r'\rho(r',t)\,\mathrm{d}r', \tag{8}$$

where β is a positive constant.

There remains to take into account the fourth factor, namely the arrival or departure of residents from the city and its environs. The simple way to do this is via the boundary conditions for the integrodifferential equation (8). Utilizing a flux argument as in descriptions of heat conduction (see, for example, Davis & Snider 1975) a suitable condition at $r = 0$ is

$$\lim_{r\to 0} r\rho_r = \alpha,$$

where α is negative if there is net immigration and positive if there is net emigration. If appropriate, α can be made a function of time or population size or distribution, though in the work which we present α will be a constant.

In addition, one needs to specify an initial distribution for the population:

$$\rho(r,0) = \phi(r), \quad r \geqslant 0,$$

and possibly a boundedness condition at $r = \infty$.

If in the above model there is no diffusion ($D = 0$), then ρ can only change at any location if there are already residents there. This case is not without interest and has the advantage that the steady-state distribution can be found exactly as will be seen below. There are other mathematical processes apart from diffusion which could be used to model the spread of population into non-occupied areas. One possibility is to insert a term for migration which depends on the existing distribution. One expects that there will be large influx of immigrants to a district which is undergoing rapid growth. A measure of this is the rate of increase of the total population,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty 2\pi r'\rho(r',t)\,\mathrm{d}r'.$$

One may modulate this factor by a density-dependent function $f(\rho)$. Then, retaining the logistic growth term and the integral inhibition term one obtains an alternative to the diffusion model (8):

$$\rho_t = k\rho(\sigma-\rho)+f(\rho)\int_0^\infty 2\pi r'\rho_t(r',t)\,\mathrm{d}r'-\beta\rho\int_0^r 2\pi r'\rho(r',t)\,\mathrm{d}r'. \tag{9}$$

This model also has interesting steady-state solutions as seen below.

3. Steady-state solutions

The model equation (8) is reminiscent of the Fitzhugh–Nagumo equation which arises as a qualitative model for the creation and propagation of a nerve impulse (see, for example, Tuckwell 1988). Here $u = u(x,t)$ satisfies the equation

$$u_t = Du_{xx} + F(u) - \gamma \int_0^t u(x,t')\,\mathrm{d}t',$$

where the integral term is in time rather than space and is called a killing term, and $F(u)$ is a cubic. We use the same terminology for our urban growth models (8) and (9). The integral term in (8) can be interpreted as reflecting the consumption of resources (roads, transport, etc.) by the population closer to the centre than the local value of r. It is similar to an integral term used in modelling the growth of certain cell populations (Bass *et al.* 1987).

To facilitate the computations which follow, we will restrict our attention for the most part to the development of a population in one space dimension, such as would be the case for growth out from a straight coastline or river. Let $x \geqslant 0$ be the distance from the reference line. The model with diffusion, growth, migration and a killing term now takes the form, with $\rho = \rho(x,t)$,

$$\rho_t = D\rho_{xx} + k\rho(\sigma - \rho) - \beta\rho \int_0^x \rho(x',t)\,\mathrm{d}x', \tag{10}$$

$$\rho(x,0) = \phi(x), \quad x \geqslant 0,$$

$$\rho_x(0,t) = \alpha.$$

Here ρ is a density per unit length. As $t \to \infty$, a steady-state population distribution will develop and this will be given by the solution of the appropriate model equation after putting the time derivative to zero. We denote the steady-state solutions by $\tilde{\rho}(x)$.

(*a*) *The case* $D = 0$

When $D = 0$ in (10), we have in the steady state,

$$k\tilde{\rho}(\sigma - \tilde{\rho}) - \beta\tilde{\rho} \int_0^x \tilde{\rho}(x')\,\mathrm{d}x' = 0.$$

On differentiating we have

$$k\,\mathrm{d}\tilde{\rho}/\mathrm{d}x + \beta\tilde{\rho} = 0,$$

whose general solution is $\tilde{\rho}(x) = \tilde{\rho}(0)\,\mathrm{e}^{-(\beta/k)x}$. However, at $x = 0$, $\rho(x,t)$ satisfies the usual logistic equation which means that $\rho(0,t) \to \sigma$ as $t \to \infty$. Hence the value of $\tilde{\rho}(0)$ is σ and we have

$$\tilde{\rho}(x) = \sigma\,\mathrm{e}^{-(\beta/k)x}.$$

Thus this model predicts an exponentially decaying population density per unit length and a total population

$$N_{\mathrm{tot}} = \int_0^\infty \tilde{\rho}(x)\,\mathrm{d}x = \frac{\sigma k}{\beta}.$$

(*b*) *The case $D = 0$ in two space dimensions*

Putting $D = 0$ in (8), setting the time derivative at zero and dividing by ρ we find that the steady-state density satisfies

$$k\,\mathrm{d}\tilde{\rho}/\mathrm{d}r + 2\pi r\beta\tilde{\rho} = 0.$$

Again ρ must grow logistically to σ at the origin, and the steady-state solution is given by

$$\tilde{\rho}(r) = \sigma\,\mathrm{e}^{-\pi(\beta/k)\,r^2}. \tag{11}$$

The population density therefore has a (semi) gaussian profile with a less pronounced peak than in the exponential case. The total population size is

$$N_{\mathrm{tot}} = \int_0^\infty \tilde{\rho}(r)\,2\pi r\,\mathrm{d}r = \frac{\sigma k}{\beta} \tag{12}$$

as before. The less pronounced peak of this gaussian profile could lead to a better fit to much of the data presented in Clark (1951), corresponding with a drop of residential density in the central business district.

(*c*) *The case $D = 0$ with migration*

If we put $\rho_t = 0$ in (9) we obtain precisely the same differential equation for $\tilde{\rho}$ as in the previous case (*b*). Hence regardless of the rate or pattern of migration, the steady state is given by (11) and the total population is given by (12). The latter depends therefore only on the parameters σ, β and k.

(*d*) *Diffusion in one space dimension*

In the case of the general model given by (10), steady-state solutions must satisfy

$$D\frac{\mathrm{d}^2\tilde{\rho}}{\mathrm{d}x^2} + k\tilde{\rho}(\sigma - \tilde{\rho}) - \beta\tilde{\rho}\int_0^x \tilde{\rho}(x')\,\mathrm{d}x' = 0. \tag{13}$$

A special solution of this equation is

$$\tilde{\rho}(x) = (\sigma + D\beta^2/k^3)\,\mathrm{e}^{-(\beta/k)\,x},$$

but in general, (13) leads to a nonlinear differential equation of third order for which solutions have to be found numerically.

It is convenient for numerical work to transform (10) and (13) by introducing dimensionless variables

$$\rho^* = \rho/\sigma, \quad \xi = (\beta/k)\,x, \quad \tau = \sigma k t$$

and the constants

$$\mu = \beta^2 D/k^3\sigma, \quad \nu = \alpha k/\beta\sigma.$$

We then have

$$\rho^*_\tau = \mu\rho^*_{\xi\xi} + \rho^*(1-\rho^*) - \rho^*\int_0^\xi \rho^*(\xi',\tau)\,\mathrm{d}\xi', \tag{10$'$}$$

$$\rho^*(\xi, 0) = \phi^*(\xi), \quad \xi \geqslant 0,$$

$$\rho^*_\xi(0,\tau) = \nu,$$

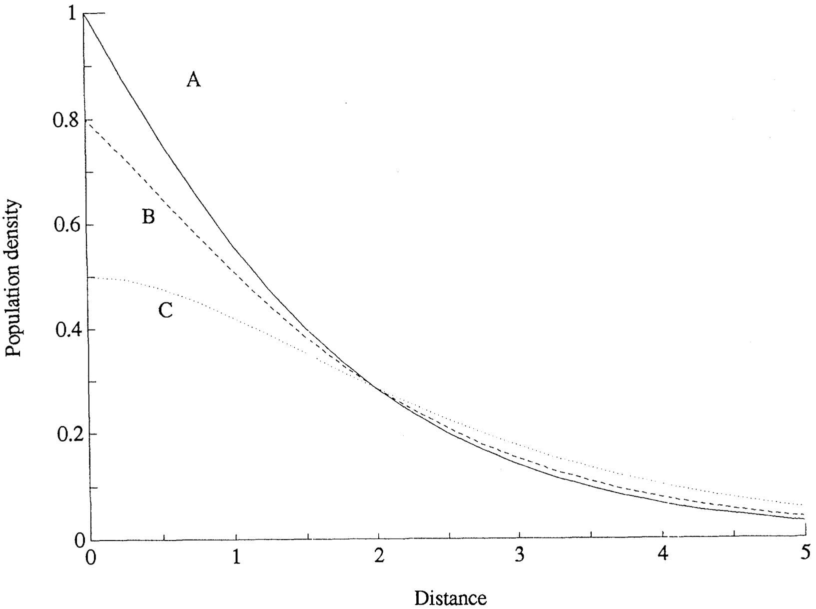

Figure 1. Computed steady-state population densities for various migration rates. For numerical values see text.

which shows that the structure of solutions is determined by initial conditions and the two dimensionless constants μ and ν. Steady-state solutions now satisfy

$$\mu\tilde{\rho}^*_{\xi\xi}+\tilde{\rho}^*(1-\tilde{\rho}^*)-\tilde{\rho}^*\int_0^{\xi}\tilde{\rho}^*(\xi')\,\mathrm{d}\xi' = 0, \tag{13'}$$

with boundary condition

$$\tilde{\rho}^*_{\xi}(0) = \nu.$$

Equation (13′) can be converted to a system of three first-order equations in which

$$y_1 = \int_0^{\xi}\tilde{\rho}^*(\xi')\,\mathrm{d}\xi'$$

and

$$\mathrm{d}y_1/\mathrm{d}\xi = y_2, \quad \mathrm{d}y_2/\mathrm{d}\xi = y_3,$$
$$\mu\,\mathrm{d}y_3/\mathrm{d}\xi = y_1 y_2 - y_2(1-y_2), \quad \xi > 0.$$

To solve this system a Runge–Kutta scheme with error control was used. We must always have $y_1(0) = 0$. This system is singular in the sense that the initial ($\xi = 0$) values for the other two components are not arbitrary, with most choices leading to unbounded solutions.

For a fixed value of μ, a given value of $\tilde{\rho}^*(0)$ will only usually give a bounded positive solution for one particular value of $\nu = \tilde{\rho}^{*\prime}(0)$. That is, if the steady state population density is to have a certain non-zero value at the origin, there is a unique migration rate which will lead to such a population distribution. A bad choice of the pair $\tilde{\rho}^*(0)$ and $\tilde{\rho}^{*\prime}(0)$ lead to solutions $\tilde{\rho}^*(\xi)$ which either go negative and then diverge to infinity, or diverge to infinity without going negative.

In figure 1 we show computed solutions of (13′) for $\mu = 1$. The values of $\tilde{\rho}^*(0)$ for the curves marked A, B, C respectively are 1.0, 0.8 and 0.5; the corresponding values of $\tilde{\rho}^{*\prime}(0)$ are -0.493384, -0.268290, and 0.006700. Each of these three computed solutions has a tail which decays in an approximately exponential fashion.

4. Time-dependent solutions

To analyse the development of a model urban population in time and space we have used equation (10′), because working in one space dimension reduces the amount of computation considerably and is expected to indicate general phenomena of interest.

Solutions of nonlinear parabolic differential equations can be obtained numerically by using finite-differencing and an extrapolated Crank–Nicolson scheme (Lees, 1967). Equation (10′), however, contains the additional feature of the killing term in which the integral was evaluated by using the trapezoidal rule. Approximating $\rho^*(i\Delta\xi, j\Delta\tau)$ by U_{ij} the basic difference equation of the numerical scheme is, in tridiagonal form,

$$-rU_{i-1,j+1}+(1+2r)\,U_{i,j+1}-rU_{i+1,j+1} = rU_{i-1,j}+(1-2r)\,U_{i,j}+rU_{i+1,j} + \Delta\tau F(\tfrac{3}{2}U_{i,j}-\tfrac{1}{2}U_{i,j-1})-sU_{i,j}\sum_{k=1}^{i}[U_{k-1,j}+U_{k,j}], \quad (14)$$

where

$$F(y) = y(1-y),$$
$$r = \mu\Delta\tau/2(\Delta\xi)^2, \quad s = \tfrac{1}{2}\Delta\tau\Delta\xi.$$

The basic difference equation is modified at boundary points and to take care of the starting conditions. Details of such modifications are discussed as well as the solution of the tridiagonal system in Tuckwell (1988). The numerical integration must be carried out on a finite interval $[0, L]$ and $L = 15$ was found to be sufficiently large. Although a Dirichlet condition could be applied at $\xi = 0$, a Neumann condition was used because it is more natural to specify an immigration rate than a population density at the origin. Although immigrants may settle anywhere, the present model has them arriving at a common point, corresponding to the centre of town, and then dispersing.

An initial population density of the form

$$\rho^*(\xi, 0) = c\exp(-(\xi-\xi_0)^2/\gamma), \quad \xi > 0,$$

was used, and in most of the numerical work the parameter values were set at $c = 0.5$, $\gamma = 0.25$, and $\xi_0 = 0$. The values of $\Delta\tau$ and $\Delta\xi$ were set at 0.005 and 0.05 respectively. The condition used at the right boundary was $\rho^*(L, \tau) = 0$.

In figure 2 are shown the computed solutions of (10′) with $\mu = 1$, with other parameters as given above except that the initial maximum population density is $c = 0.6$. The boundary condition at the origin is $\rho^*_\xi(0, \tau) = -0.2682895$ as in case B of figure 1. Results are shown for $\tau = 0$, $\tau = 0.5$, $\tau = 2$, and $\tau = 10$. The initial distribution at first decreases near $\xi = 0$ and spreads. This is followed by growth and spread until the steady state is approached. The steady-state distribution (taken as that at $\tau = 10$ after observing no change for a long time interval) was exactly (to four or more significant figures) the same as that obtained as the unique bounded numerical solution (B) of the steady-state ordinary differential equation (13′) with the same parameter values. This was a very useful check on the accuracy of the numerical schemes used for both the steady-state and time-dependent problems. By

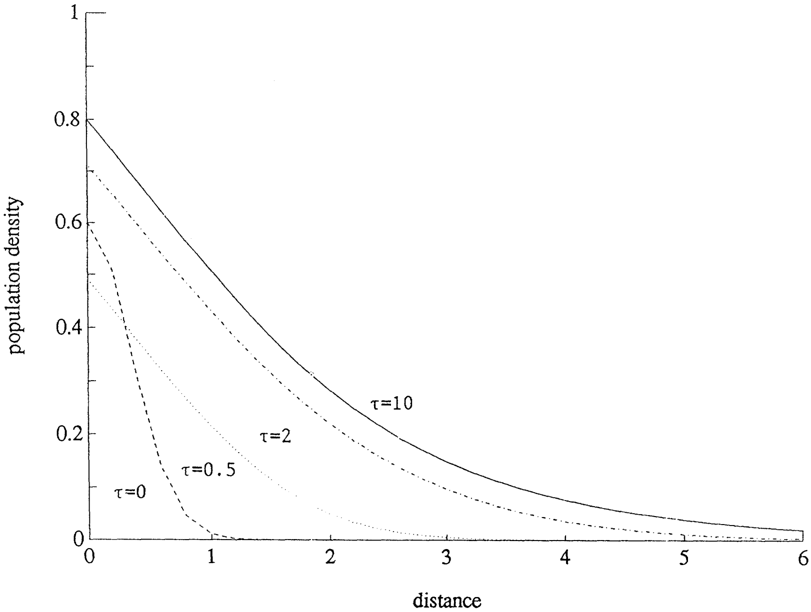

Figure 2. Time course of the population density during evolution to a steady state.

numerical integration, the total populations at $\tau = 0.5$, $\tau = 2$, and $\tau = 10$ were found to be 0.4947, 1.1365 and 1.4765, respectively. Furthermore, when the initial population density was altered, for example by changing the initial maximum to $c = 2.0$, the steady state solution was precisely the same. In this case the population density declined at the origin whilst the distribution of residents became scattered over a continually wider area, as is the case for the data of table 1 for London. Furthermore, calculated solutions of (10′) with $\rho^*_\xi(0,\tau) = -0.493\,384$ and $\rho^*_\xi(0,\tau) = 0.006\,700$ tended precisely to the results depicted in curves A and C,m respectively, in figure 1.

Finally, we consider some of the factors that determine whether a city will persist or become extinct. In particular we endeavour firstly to ascertain the effects of different migration rates when all other parameter values, including those of the initial distribution, are held fixed.

We computed, therefore, solutions of (10′) by using the same parameter values as previously, except with $c = 0.5$, and with various values of $\rho^*_\xi(0,\tau)$ corresponding to various migration rates. It was found that if the emigration rate exceeded 0.14, a city could not be sustained. That is, the population became totally extinct, because the increases due to natural (logistic) growth could not overcome the effect of too high a net emigration rate. However, for all emigration rates between 0 and 0.14, a non-zero steady-state urban population developed. Whenever there was net immigration, the city grew and persisted. These results are summarized in figure 3 where final population density at the origin and total population size are plotted against $\rho^*_\xi(0,\tau)$. Note that here it is assumed the migration rate was constant; clearly if the net flux of residents was made a function of time or population size, the results could be quite different.

Secondly, we consider the effects of altering the initial distribution of residents. We have noted that when the value of $\rho^*_\xi(0,\tau)$ was 0.15 and the constant c of the initial distribution was 0.5, a population centre failed to become established. However,

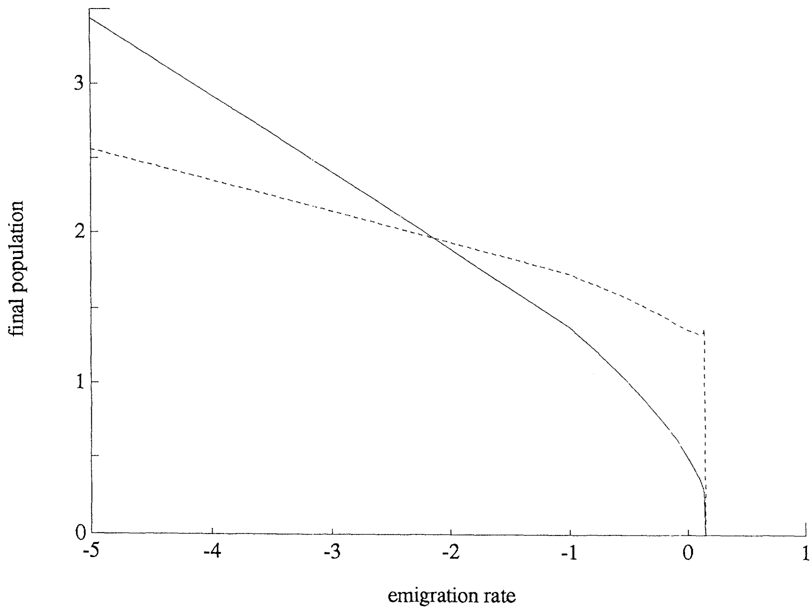

Figure 3. Characteristics of the final population distribution as the migration rate varies. ——, Density at origin; ---, total population.

doubling the value of c leads to the formation of a non-trivial equilibrium solution. Thus there is a *threshold effect*: for certain emigration rates there must be sufficient initial numbers of residents or the entire population will become extinct. Furthermore, when the emigration rate was increased to 1, an equilibrium density failed to establish itself no matter how large an initial distribution was used.

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