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Random Perturbations of the Reduced Fitzhugh–Nagumo Equation

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Abstract

Perturbations of the reduced Fitzhugh–Nagumo or Ginzburg–Landau equation by two-parameter white noise are considered via Green's functions and stochastic integral representations. Statistical properties of the solution are obtained to order ε^2. Perturbations about the lower stable point reveal that the mean is increased by zero mean white noise, whereas when the upper stable equilibrium point is considered, the mean is lowered. An explanation is sought in terms of the properties of the nonlinear forcing term near the equilibrium point.

1. Introduction

There has been much interest recently in noise-driven nonlinear systems of reaction–diffusion equations in both the physical and biological sciences. One area of physics where such systems have arisen is the stochastic quantization of field theories [1–4] where the correlation functions of solutions can be employed to evaluate amplitudes associated with Feynman graphs. Additionally such systems have arisen in statistical mechanics where a time-dependent Langevin equation is used in the Ginzburg–Landau theory of phase transitions [5, 6].

Similar stochastic partial differential equations have arisen in neurophysiology. The well-known Hodgkin–Huxley equation [7] are a set of four nonlinear partial differential equations with diffusion in the first equation, that for the transmembrane electrical potential. The remaining three equations describe the behaviour of the conductances to various ions and under certain conditions there will form a solitary wave or action potential [8].

Consideration of random perturbations in the Hodgkin–Huxley equations is of interest to study the quantitative effects of random input currents in neurons. The sources of the random currents can be synaptic, of excitatory and inhibitory natures, or due to the random opening and closing of ion channels. Some methods of determining such results for multidimensional systems have been given [9] but the ensuing calculations are very long. One therefore turns to simpler systems of equations whose solutions mimic those of the Hodgkin–Huxley system. A two-component system with such properties is that of Fitzhugh–Nagumo [10]. A further simplification results on setting to zero one of the parameters which controls recovery to resting level. The scalar equation that results has two stable states and a superthreshold stimulus leads to the formation of a saturating travelling wave. Such an equation is called the reduced Fitzhugh–Nagumo equation. The same partial differential equation is called a Ginzburg–Landau equation in statistical mechanics [5].

More generally, let $u(x, t)$ be a two-parameter random process formally satisfying the parabolic equation

$$u_t = u_{xx} + g(u) + \varepsilon(\alpha + \beta W_{xt}) \tag{1}$$

where $0 < x < L$, $t > 0$; $u(x, 0) = 0$ wp 1, $0 < x < L$; $u_x(0, t) = u_x(L, t) = 0$, $t > 0$; $\{W(x, t)\}$ is a standard two-parameter Wiener process, g is a non-random real-valued function with the required smoothness properties. It is assumed that the system has at least one asymptotically stable equilibrium point. If the mixed second partial derivative W_{xt}, which one formally calls two-parameter white noise, is noted by $w(x, t)$, then we have

$$W(x, t) = \int_0^x \int_0^t w(y, s)\, \mathrm{d}s\, \mathrm{d}y$$

and w is a Gaussian process with zero mean and correlation function

$$E[w(x, t)w(y, s)] = \delta(x - y)\delta(t - s)$$

In the reduced Fitzhugh–Nagumo case,

$$g(u) = u(u - a)(1 - u) \tag{2}$$

where $0 < a < 1$. We are interested in calculating the statistical properties of solutions of eq. (1) when the system is perturbed about an equilibrium point. We will see that there is an interesting difference between the cases of perturbing about upper and lower stable equilibrium points of the reduced Fitzhugh–Nagumo system. When random fluctuations drive the system about the lower stable point $u_0 = 0$, the mean of $u(x, t)$ is displaced upward even when the constant α is zero so that the added noise has zero mean. Surprisingly, random fluctuations about the upper stable point $u_0 = 1$ cause the mean to be displaced downwards.

2. A perturbative expansion

Since we are interested in fluctuations about an equilibrium point u_0 at which $g(u) = 0$, we put

$$u = u_0 + \sum_{k=1}^{\infty} \varepsilon^k u_k \tag{3}$$

and we require that u_0 is a stable point, so that $f'(u_0) < 0$. The convergence of such expansions has been investigated [11] as well as existence and uniqueness of the solution [12]. Substituting in eq. (1) and equating coefficients of powers of ε, we find that the terms in eq. (3) satisfy a sequence of linear stochastic partial differential equations whose forcing terms are either given or are known solutions of preceding equations:

$$u_1(x, t) = \int_0^t \int_0^L G(x, y; t-s)[\alpha\, dy\, ds + \beta\, dW(y, s)] \quad (4a)$$

$$u_2(x, t) = \frac{g''(u_0)}{2} \int_0^t \int_0^L G(x, y; t-s) u_1^2(y, s)\, dy\, ds \quad (4b)$$

$$u_3(x, t) = g''(u_0) \int_0^t \int_0^L G(x, y; t-s) u_1(y, s) u_2(y, s)\, dy\, ds + \frac{g'''(u_0)}{3!} \int_0^t \int_0^L G(x, y; t-s) u_1^3(y, s)\, dy\, ds \quad (4c)$$

where G is the (deterministic) Green's function for the modified heat equation

$$u_t = u_{xx} + g'(u_0)u \quad (5)$$

with appropriate boundary conditions.

When $|\alpha| > 0$, the positions and possibly the natures of the equilibrium points will change. It is then of interest to ascertain whether there may be noise-induced (phase) transitions to excited states which could not have occurred in the absence of noise. The expectation of u_1 is given by

$$E[u_1(x, t)] = \frac{\alpha}{g'(u_0)} (e^{g'(u_0)t} - 1) \quad (6)$$

but the determination of $E[u_2]$ is not so straightforward. Let us put

$$E[u_2] = E[u_2]_{(1)} + E[u_2]_{(2)} \quad (7)$$

where the first and simpler contribution is defined by

$$E[u_2(x, t)]_{(1)} = \frac{g''(u_0)}{2!} \int_0^t \int_0^L G(x, y; t-s) E^2[u_1(y, s)]\, dy\, ds \quad (8)$$

Using the appropriate eigenfunction expansion for G,

$$G(x, y; t) = \frac{e^{g'(u_0)t}}{L} \left(1 + 2 \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y)\, e^{-\mu_n t}\right),$$

where

$$\phi_n(x) = \cos(n\pi x/L),$$

$$\mu_n = \frac{n^2\pi^2}{L^2},$$

from eq. (8) we get

$$E[u_2(x, t)]_{(1)} = \frac{g''(u_0)\alpha^2}{2L[g'(u_0)]^2} \int_0^t \int_0^L e^{g'(u_0)(t-s)} \times \left\{1 + 2 \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y)\, e^{-\mu_n(t-s)}\right\} \times \{e^{2g'(u_0)s} - 2\, e^{g'(u_0)s} + 1\}\, dy\, ds$$

which simplifies to

$$E[u_2(x, t)]_{(1)} = \frac{\alpha^2 g''(u_0)\, e^{-g'(u_0)t}}{g'(u_0)^2} \left[\frac{\sinh[g'(u_0)t]}{g'(u_0)} - t\right] \quad (9)$$

The remaining contribution to $E[u_2]$ is

$$E[u_2(x, t)]_{(2)} = \frac{g''(u_0)}{2} \int_0^t \int_0^L G(x, y; t-s) \times \mathrm{Var}\,[u_1(y, s)]\, dy\, ds,$$

and, since $\mathrm{Var}\,[u_1(y, s)] < \infty$ for all $y \in [0, L]$ and for all $s \geqslant 0$, this contribution is finite. To evaluate it we need to find

$$\frac{\beta^2}{L^2} \int_0^t \int_0^L e^{g'(u_0)(t-s)} \left[1 + 2 \sum_{m=1}^{\infty} \phi_m(x)\phi_m(y)\, e^{-\mu_m(t-s)}\right] \times \left(\frac{\{e^{2g'(u_0)s} - 1\}}{2g'(u_0)} + \sum_{n=1}^{\infty} \frac{\phi_n^2\{1 - e^{2s[g'(u_0) - \mu_n]}\}}{\mu_n - g'(u_0)}\right) dy\, ds$$

which, with $\gamma = g'(u_0)$, leads finally to

$$E[u_2(x, t)]_{(2)} = [g''(u_0)\beta^2/4L] \times \left([(e^{\gamma t} - 1)/\gamma]^2 + \sum_{n=1}^{\infty} [A_n(t) + B_n(x, t)]\right)$$

where

$$A_n(t) = \frac{1}{\mu_n - \gamma} \left(\frac{e^{\gamma t} - 1}{\gamma} + \frac{e^{\gamma t} - e^{2t(\gamma - \mu_n)}}{\gamma - 2\mu_n}\right)$$

and where

$$B_n(x, t) = \frac{\phi_{2n}(x)}{\mu_n - \gamma} \left(\frac{1 - e^{t(\gamma - \mu_{2n})}}{\mu_{2n} - \gamma} + \frac{e^{t(\gamma - \mu_{2n})} - e^{2t(\gamma - \mu_n)}}{\gamma + \mu_{2n} - 2\mu_n}\right)$$

This gives the expectation of $u(x, t)$ to order ε^2. In the case of the reduced Fitzhugh–Nagumo system,

$$E[u(x, t)] = u_0 + \frac{\varepsilon\alpha}{\gamma} (e^{\gamma t} - 1) + \varepsilon^2 E[u_2(x, t)] + O(\varepsilon^3) \cdots$$

Higher order moments, including covariance, and spectral density have also been determined to order ε^2 but the expressions, which are very lengthy, will not be displayed here.

3. Effect of noise at the equilibrium points

From the above results, an expression for the steady state mean can be obtained. Some very interesting and perhaps unexpected results emerge immediately. Choosing $u_0 = 0$ and $\alpha = 0$ so that the forcing term is purely random with zero mean (i.e., the deterministic component is zero), we have in the reduced Fitzhugh–Nagumo case as $t \to \infty$:

$$E[u(x, t)] \to \frac{\beta^2\varepsilon^2(1 + a)}{2L} \left(\frac{1}{a^2} + \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{\mu_n + a} + \sum_{n=1}^{\infty} \frac{\phi_{2n}(x)}{(\mu_n + a)(\mu_{2n} + a)}\right) + O(\varepsilon^3). \quad (10)$$

We can see that the $O(\varepsilon^2)$ term here is positive. This follows because

$$\frac{1}{a^2} > 0, \quad \frac{1}{a} \sum \frac{1}{\mu_n + a} > 0$$

and

$$\left|\sum_{n=1}^{\infty} \frac{\phi_{2n}(x)}{(\mu_n + a)(\mu_{2n} + a)}\right| < \frac{1}{a} \left|\sum_{n=1}^{\infty} \frac{\phi_{2n}(x)}{\mu_n + a}\right| < \frac{1}{a} \left|\sum_{n=1}^{\infty} \frac{1}{\mu_n + a}\right|.$$

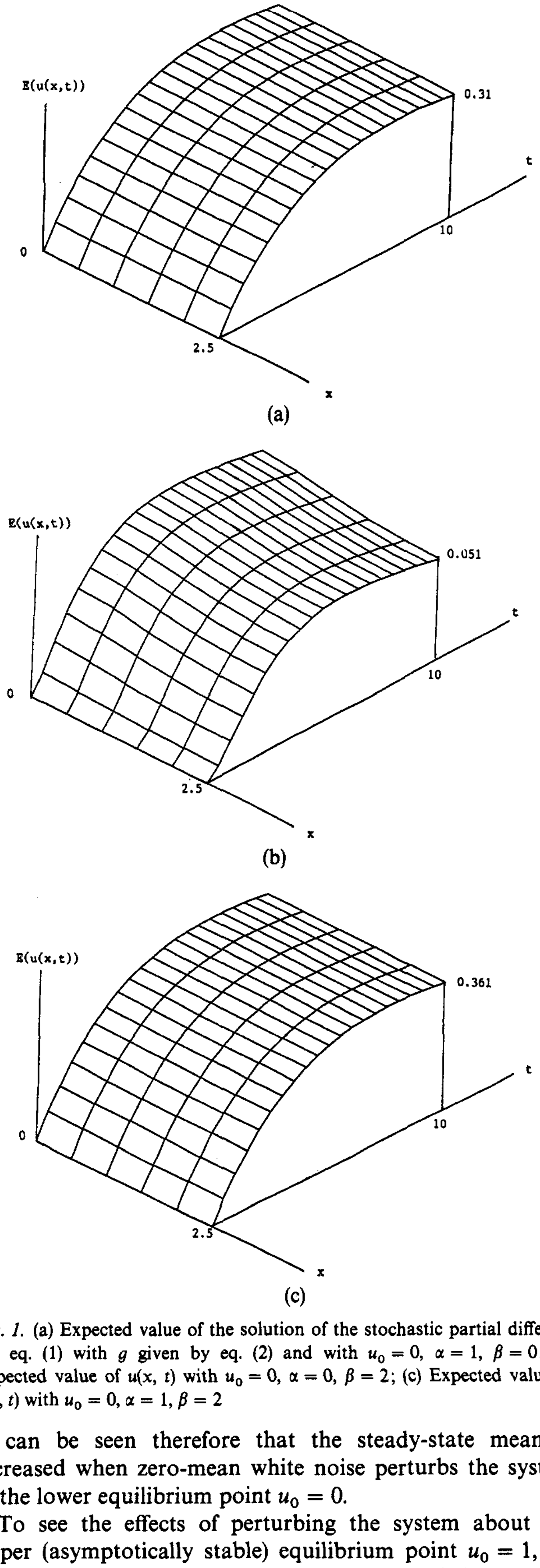

Fig. 1. (a) Expected value of the solution of the stochastic partial differential eq. (1) with g given by eq. (2) and with $u_0 = 0$, $\alpha = 1$, $\beta = 0$; (b) Expected value of $u(x, t)$ with $u_0 = 0$, $\alpha = 0$, $\beta = 2$; (c) Expected value of $u(x, t)$ with $u_0 = 0$, $\alpha = 1$, $\beta = 2$

It can be seen therefore that the steady-state mean is increased when zero-mean white noise perturbs the system at the lower equilibrium point $u_0 = 0$.

To see the effects of perturbing the system about the upper (asymptotically stable) equilibrium point $u_0 = 1$, we make the change of variable $v = 1 - u$. Then it is apparent that the solution of the system

$$v_t = v_{xx} + v(v - a)(1 - v) + \varepsilon(\alpha + \beta W_{xt})$$

$$v(x, 0) = 1$$

$$v_x(0, t) = v_x(L, t) = 0$$

is obtained from that of eq. (1) [with $g(u)$ given by eq. (2)], by making the substitutions $a \to 1 - a$, $\alpha \to -\alpha$, $\beta \to -\beta$, $v = 1 - u$ and $E[v] = 1 - E[u]$. In the case $u_0 = 1$, as $t \to \infty$,

$$E[u(x, t)] \to 1 - \frac{\beta^2\varepsilon^2(2 - a)}{2L} \times \left(\frac{1}{(1 - a)^2} + \frac{1}{(1 - a)} \sum_{n=1}^{\infty} \frac{1}{\mu_n + (1 - a)} + \sum_{n=1}^{\infty} \frac{\phi_{2n}(x)}{(\mu_n + 1 - a)(\mu_{2n} + 1 - a)}\right) + O(\varepsilon^3) \quad (11)$$

Using the same arguments which were used to show that the $O(\varepsilon^2)$ term was positive in the $u_0 = 0$ case, we draw the same conclusion here. Thus, to order ε, $E[u(x, t)] < 1$ and we see that the effect of noise on the system near the upper equilibrium point is to decrease the expected value of $u(x, t)$ for all x.

Figure 1(a–c) shows computed results for the time- and space-dependent mean to order ε^2 for the system defined by eqs (1) and (2). Here $u_0 = 0$, $u(x, 0) = 0$, $L = 2.5$, $a = 0.5$ and $\varepsilon = 0.1$. In Fig. 1(a), there is no noise as we have put $\beta = 0$ and set $\alpha = 1$. The mean steadily increases to about 0.3 at all x. In Fig. 1(b) are shown the results when there is noise only, with no deterministic input whatsoever, with $\alpha = 0$ and $\beta = 2$. The mean increases steadily again, but this time reaches an asymptotic value of about 0.05 for all x; this is the value predicted by the asymptotic expression (10). Figure 1(c) indicates the evolution of the mean when there is both noise and a deterministic forcing term. In Fig. 2 the mean is shown for the case $u_0 = 1$ with an initial value $u(x, 0) = 1$ for all x. The mean steadily decreases everywhere to reach a steady state value of about 0.95 as predicted by eq. (11). In future work we will attempt to ascertain the general principles involved in the direction of shift of the mean as a function of properties of the nonlinear forcing term near its critical points. It seems that g'' may be crucial in determining the direction of the shift, and we are currently investigating the related problem for stochastic ordinary differential equations of type

$$\mathrm{d}X = g(X)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W$$

which have been energetically studied [13, 14]. The results are expected to have important ramifications in both

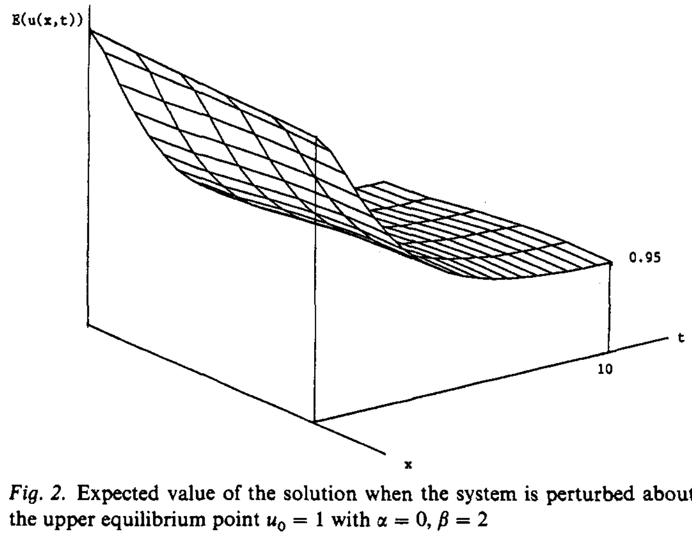

Fig. 2. Expected value of the solution when the system is perturbed about the upper equilibrium point $u_0 = 1$ with $\alpha = 0$, $\beta = 2$

biology and physics, especially where measurements are made in the presence of noise.

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