Physica Scripta. Vol. 37, 321–322, 1988.

Perturbative Analysis of Random Nonlinear Reaction-Diffusion Systems

Henry C. Tuckwell

Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia

Received April 23, 1987; accepted June 10, 1987

Abstract

A method of perturbative analysis of a class of stochastic nonlinear reaction–diffusion systems is described which employs representations of solutions as stochastic integrals. Specifically several properties of a Fitzhugh–Nagumo system perturbed by two-parameter Gaussian white noise are obtained.

Systems of nonlinear reaction–diffusion systems arise in several fields [1, 4, 5, 10, 11, 13] and have been the subject of recent analysis [3, 7]. It is important to analyze such systems not only in the deterministic case but also in the presence of random forcing terms. One area in which nonlinear stochastic partial differential equations have arisen is in the stochastic quantization of field theories [2, 6, 9] but there are many other applications such as the theory of nerve membrane potential [5] and of nonlinear waves in brain structures [11, 13]. This paper is a first step in the analysis of random nonlinear reaction–diffusion systems. It extends our previously reported work on the scalar case [12] and contains a novel use of Green's function matrices for multidimensional partial differential equations. Our method of analysis introduces representations of solutions in terms of stochastic integrals with respect to two-parameter Wiener processes.

We consider equations of the form

$$\boldsymbol{u}_t = \boldsymbol{D}\boldsymbol{u}_{xx} + \boldsymbol{g}(\boldsymbol{u}) + \varepsilon \boldsymbol{f}(x, t), \quad a < x < b, \quad t > 0, \tag{1}$$

where $\boldsymbol{u}(x, t)$ is an n-dimensional vector, $\boldsymbol{D}$ is an $n \times n$ constant diagonal matrix of diffusion coefficients, subscripts t and x represent partial derivatives with respect to these variables, $\boldsymbol{g}$ is a given n-component function of $\boldsymbol{u}$, $\boldsymbol{f}$ is a given random (or deterministic) forcing term and ε is a small parameter. In conjunction with eq. (1) suitable boundary-initial values are assumed given.

We write the solution as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sum_{k=1}^{\infty} \varepsilon^k \boldsymbol{u}_k \tag{2}$$

where $\boldsymbol{g}(\boldsymbol{u}_0) = \boldsymbol{0}$ which in eq. (1) gives a recursive sequence of systems of linear stochastic partial differential equations. The solutions of these may be found using Green's function matrices and stochastic integrals. We will demonstrate with a specific example.

The Fitzhugh–Nagumo system $\boldsymbol{u} = (u \quad v)^{\mathrm{T}}$ with diffusion in both components and additive two-parameter Gaussian white noise in one component is written

$$\begin{aligned} u_t &= u_{xx} + u(1-u)(u-a) - v + \varepsilon(\alpha + \beta W_{xt}) \\ v_t &= v_{xx} + b(u - \gamma v) \end{aligned} \tag{3}$$

where $0 < x < L < \infty$, $t > 0$, $0 < a < 1$; α, β, γ, b are constants and $\{W(x, t), 0 \leqslant x \leqslant L, t \geqslant 0\}$ is a standard 2-parameter Wiener process whose formal derivative $\{W_{xt}\}$ is a 2-parameter Gaussian white noise.

With $\boldsymbol{u}_0 = \boldsymbol{0}$ and

$$u = \sum_{k=1}^{\infty} \varepsilon^k u_k, \quad v = \sum_{k=1}^{\infty} \varepsilon^k v_k \tag{4}$$

we introduce

$$\boldsymbol{u}_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} -a & -1 \\ b & -b\gamma \end{pmatrix}. \tag{5}$$

Then we obtain the systems of linear stochastic partial differential equations

$$\boldsymbol{u}_{k,t} = \boldsymbol{u}_{k,xx} + \boldsymbol{A}\boldsymbol{u}_k + \boldsymbol{f}_k, \quad k = 1, 2, \ldots \tag{6}$$

where the first three $\boldsymbol{f}_k$'s are

$$\begin{aligned} \boldsymbol{f}_1^{\mathrm{T}} &= [\alpha + \beta W_{xt}, 0], \quad \boldsymbol{f}_2^{\mathrm{T}} = [(1 + a)u_1^2, 0], \\ \boldsymbol{f}_3^{\mathrm{T}} &= [2(1 + a)u_1 u_2, 0]. \end{aligned} \tag{7}$$

Letting $\boldsymbol{G}(x, y; t)$ be the Green's function matrix for $\boldsymbol{u}_t = \boldsymbol{u}_{xx} + \boldsymbol{A}\boldsymbol{u}$, and assuming u and v are initially zero, we find

$$\boldsymbol{u}_1(x, t) = \int_0^L \int_0^t \boldsymbol{G}(x, y; t - s)\boldsymbol{f}_1(y, s)\, \mathrm{d}s\, \mathrm{d}y. \tag{8}$$

In the present example $\boldsymbol{G}(x, y; t) = \exp(\boldsymbol{A}t)G(x, y; t)$ where G is the Green's function for the scalar heat equation $u_t = u_{xx}$. With Dirichlet conditions at 0 and L we may write

$$G(x, y; t) = (2/L) \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y) \exp(-\mu_n^2 t), \quad t > 0, \tag{9}$$

where $\phi_n(x) = \sin(n\pi x/L)$ and $\mu_n = n\pi/L$. We denote the eigenvalues of $\boldsymbol{A}$ by $\lambda_1 = -a + v/2$, $\lambda_2 = -b\gamma - v/2$, where $v = a - b\gamma - \sqrt{(a - b\gamma)^2 - 4b}$.

We take expectations in eq. (8) to obtain the $0(\varepsilon)$ term in $E[\boldsymbol{u}(x, t)]$. The covariance of $u_1(x, s)$ and $u_2(y, t)$ has been found using the stochastic integral [14] representation

$$\begin{aligned} E\{u_1(x, t) - E[u_1(x, t)]\} &= \beta \int_0^L \int_0^t [B_{11}(t)B_{11}(-s) \\ &\quad + B_{12}(t)B_{21}(-s)]G(x, y; t - s)\, \mathrm{d}W(y, s), \end{aligned} \tag{10}$$

where $B_{ij}(t)$ is the (i, j)-element of $\exp(\boldsymbol{A}t)$. This gives the variance of $u_1(x, t)$ and hence the expectation of $\boldsymbol{u}_2(x, t)$ from

$$\boldsymbol{u}_2(x, t) = \int_0^L \int_0^t \mathrm{e}^{\boldsymbol{A}(t-s)}\, G(x, y; t - s) \times \begin{pmatrix} (1 + a)u_1^2(y, s) \\ 0 \end{pmatrix} \mathrm{d}s\, \mathrm{d}y. \tag{11}$$

To determine the covariance of $u(x, s)$ and $u(y, t)$ to order ε^3 requires both Cov $[u_1(x, s), u_2(y, t)]$ and Cov $[u_2(x, s), u_1(y, t)]$. We have

$$\begin{aligned}\text{Cov}\,[u_1(x, s), u_2(y, t)] &= 2(1 + a)\alpha\beta^2 \\ &\times E\left[\left\{\int_0^L \int_0^s F(s, s_1)G(x, x_1; s - s_1)\,\mathrm{d}W(x_1, s_1)\right\}\right. \\ &\times \left\{\int_0^L \int_0^t G(y, y_1; t - t_1)F(t, t_1)\right. \\ &\times \int_0^L \int_0^{t_1} F(t_1, t_2)G(y_1, y_2; t_1 - t_2)\,\mathrm{d}t_2\,\mathrm{d}y_2 \\ &\times \left.\left.\left(\int_0^{t_1}\int_0^L F(t_1, t_3)G(y_1, y_3; t_1 - t_3)\,\mathrm{d}W(y_3, t_3)\right)\mathrm{d}y_1\,\mathrm{d}t_1\right\}\right]\end{aligned}$$

where $F(t, t_1) = B_{11}(t)B_{11}(-t_1) + B_{12}(t)B_{21}(-t_1)$. This covariance has been found exactly and hence the variance to order ε^3. The asymptotic $t \to \infty$ expression for the variance is

$$\begin{aligned}\text{Var}\,[u(x, \infty)] &= k_1\varepsilon^2 \sum_{n=1}^{\infty} \phi_n^2(x)(\alpha_n + \beta_n + \gamma_n) \\ &+ 2k_2\varepsilon^3 \sum_{l=1}^{\infty}{}' \sum_{m=1}^{\infty}{}' \sum_{n=1}^{\infty}{}' V_{lmn}(x)\left\{\lambda\left(\frac{\alpha_n + \beta_n/2}{\mu_l^2 + 2\mu_n^2 - 2\lambda_1}\right)\right. \\ &+ \mu\left(\frac{\gamma_n + \beta_n/2}{\mu_l^2 + 2\mu_n^2 - 2\lambda_2}\right) \\ &+ \left.\frac{\lambda(\gamma_n + \beta_n/2) + \mu(\alpha_n + \beta_n/2)}{\mu_l^2 + 2\mu_n^2 - \lambda_1 - \lambda_2}\right\} + 0(\varepsilon^4)\end{aligned}$$

where

$$\begin{aligned}\lambda &= b\gamma + v/2 - \alpha, \quad \mu = v/2, \quad \alpha_n = \lambda^2/2(\mu_n^2 - \lambda_1), \\ \beta_n &= 2\lambda\mu/(2\mu_n^2 - \lambda_1 - \lambda_2), \quad \gamma_n = \mu^2/2(\mu_n^2 - \lambda_2), \\ k_1 &= 2\beta^2(\lambda + \mu)^2/(b\gamma + v - a)^4 L, \\ k_2 &= 8(1 + \alpha)\alpha\beta^2(\lambda + \mu)^3/(b\gamma + v - a)^8\pi^2 L, \\ V_{lmn}(x) &= U_{lmn}\phi_l(x)P_m\phi_n(x)/(2m - 1),\end{aligned}$$

U_{lmn} and P_m being constants. The primes on the summations indicate that not all terms are present. Similarly an expression has been obtained for the covariance of $u(x, s)$ and $u(y, t)$ to order ε^3. The spectral density of the asymptotically weakly stationary process $\{u(x, t)\}$ is given by

$$\begin{aligned}f_u(\omega; x) &= (k_1\varepsilon^2/\pi)\sum_{n=1}^{\infty}\phi_n^2(x)\left\{\frac{(\alpha_n + \beta_n/2)(|\lambda_1| + \mu_n^2)}{(|\lambda_1| + \mu_n^2)^2 + \omega^2}\right. \\ &+ \left.\frac{(\gamma_n + \beta_n/2)(|\lambda_2| + \mu_n^2)}{(|\lambda_2| + \mu_n^2)^2 + \omega^2}\right\} + (k_2\varepsilon^3/\pi)\sum_{l=1}^{\infty}{}'\sum_{m=1}^{\infty}{}'\sum_{n=1}^{\infty}{}' V_{lmn}(x) \\ &\times \left\{\frac{(a_{nl} + e_{nl})(|\lambda_1| + \mu_n^2)}{(|\lambda_1| + \mu_n^2)^2 + \omega^2} + \frac{(b_{nl} + f_{nl})(|\lambda_2| + \mu_n^2)}{(|\lambda_2| + \mu_n^2)^2 + \omega^2}\right. \\ &+ \left.\frac{c_{nl}(|\lambda_1| + \mu_l^2 + \mu_n^2)}{(|\lambda_1| + \mu_l^2 + \mu_n^2)^2 + \omega^2} + \frac{d_{nl}(|\lambda_2| + \mu_l^2 + \mu_n^2)}{(|\lambda_2| + \mu_l^2 + \mu_n^2)^2 + \omega^2}\right\} \\ &+ 0(\varepsilon^4),\end{aligned}$$

where $a_{nl}, \ldots, f_{nl}$ are constants. Thus the effects of the non-linear terms on the moments and spectral density have been found which is expected to be relevant to theories of $1/f$ noise. One main question concerns the effect of noise on wave propagation. Whereas a deterministic reaction–diffusion system may support solitary waves [5, 7, 8, 10, 11, 13] under random perturbations an exact solitary wave cannot exist. The moments of the components, however, may have solitary wave properties. Details of the above calculations and numerical solutions will be presented in a forthcoming paper. These results are a first step towards an understanding and analysis of more complicated systems such as the Hodgkin-Huxley equations perturbed by noise.

References

1. Aronson, D. and Weinberger, H., Adv. Math. **30**, 33 (1978).
2. Floratos, E. and Iliopoulos, J., Nucl. Phys. **B214**, 392 (1983).
3. Freidlin, M., Ann. Prob. **13**, 639 (1985).
4. Gierer, A. and Meinhardt, H., Kybernetik **12**, 30 (1972).
5. Hodgkin, A. L. and Huxley, A. F., J. Physiol. **117**, 500 (1952).
6. Jona-Lasinio, G. and Mitter, P. K., Comm. Math. Phys. **101**, 409 (1985).
7. Jones, C. K. R. T., Trans. Amer. Math. Soc. **286**, 431 (1984).
8. Nagumo, J. S., Arimoto, S. and Yoshizawa, S., Proc. I.R.E. **50**, 2061 (1962).
9. Parisi, G. and Wu, Y., Sci. Sin. **24**, 483 (1981).
10. Smoller, J., Shock Waves and Reaction-Diffusion Equations, Springer, New York (1983).
11. Tuckwell, H. C., Int. J. Neurosci. **12**, 95 (1981).
12. Tuckwell, H. C., Phys. Lett. A **122**, 117 (1987).
13. Tuckwell, H. C. and Miura, R. M., Biophys. J. **23**, 257 (1978).
14. Wong, E. and Zakai, M., Z. Wahrschreinlicheitsth. **29**, 109 (1974).