Nonlinear diffusions on bounded intervals perturbed by gaussian white noise are considered. Terms in the expansion of the solution satisfy a recursive sequence of linear stochastic partial differential equations with the same kernel. Their solutions may be found as stochastic integrals. Thus expressions are obtained for the mean to order $\epsilon^2$ and for the covariance and spectral density to order $\epsilon^3$.

Nonlinear diffusion equations arise in diverse areas including chemistry [1], genetics [2], neurobiology [3] and physics [4]. Recently there has been much interest in such equations with additive gaussian white noise. One source of impetus for such studies has been the procedure of stochastic quantization of field theories initiated by Parisi and Wu [5]. In that program, stochastic partial differential equations are considered of the form

$$\phi_t = -\delta S/\delta \phi + w(x, t), \quad (1)$$

where $\phi(x, t)$ is a field, $x \in \mathbb{R}^3$, $S$ is an action functional and $w(x, t)$ is a gaussian white noise. (Throughout, the subscripts $t$ and $x$ indicate partial derivatives with respect to these variables.) The parameter $\lambda$ is an artificially introduced one because usually only the invariant ($\lambda = \infty$) distributions are of interest. In particular, the correlation functions

$$\lim_{t \to \infty} E(\phi(x_1, t) \ldots \phi(x_n, t)), \quad n = 1, 2, \ldots,$$

are sought as these are the propagators used in evaluating amplitude contributions from Feynman diagrams.

This approach has advantages for gauge theories [5] and has been established for perturbative theories of gauge invariant and non-gauge invariant type [6]. Nonperturbative bounds have been found for $E(\phi^n)$ using a band-limited version of white noise in a $\lambda \phi^4$ field theory [7]. A rigorous nonperturbative analysis has recently appeared for $P(\phi)^2$ field theory on bounded domains [8].

Similar nonlinear stochastic partial differential equations arise in the theory of nerve membrane potential [9] and are pertinent to theories of 1/f noise. If $\phi$ represents the depolarization, then in the presence of random currents we may employ the equation

$$\phi_t = \phi_{xx} + f(\phi) + \epsilon (\alpha + \beta w) \quad (2)$$

to describe the behaviour of the potential in a one-component approximation. In (2), $\epsilon > 0$, $\alpha$ and $\beta$ are constants and $f$ is a possibly nonlinear function. Equations of this type were previously considered and asymptotic expansions for $\epsilon \to 0$ and Lipschitz continuous $f$ were investigated [10].

As in ref. [11] we are interested in solutions on finite intervals and here restrict our attention to one space dimension. In higher dimensions solutions exist in Sobolev spaces of distributions [8]. We thus take $w$ as a
two-parameter gaussian white noise and note that in the linear case $f(\phi) = -\phi$ many of the solution properties are known [10–12] especially as Fourier transforms [5]. An example of our approach can be illustrated with the reduced FitzHugh–Nagumo equation driven by white noise:

$$\phi_t = \phi_{xx} + \phi(\phi - a)(1 - \phi) + \epsilon(\alpha + \beta W_{xt})$$

(3)

on $0 < x < L$, $t > 0$ with $\phi(x, 0) = 0$, $0 \leq x \leq L$ and with Neumann conditions $\phi_x(0, t) = \phi_x(L, t) = 0$, $t \geq 0$. Here $\{W(x, t), x \in [0, L], t \geq 0\}$ is a standard two-parameter Wiener process and $0 < a < 1$. Note that if $w(x, t)$ is a two-parameter white noise then the relation between $w$ and $W$ may be formally written $w(x, t) = \delta^2 W/\delta x \delta t$ and that $W(x, t)$ has mean 0 and variance $xt$.

Assuming $f(\phi_t) = 0$ we put

$$\phi = \phi_0 + \sum_{k=1}^{\infty} \epsilon^k \phi_k,$$

(4)

which on substitution in (3) gives

$$\phi_{1,t} = \phi_{1,xx} + f'(\phi_0) \phi_1 + \alpha + \beta W_{xt},$$

(5)

$$\phi_{2,t} = \phi_{2,xx} + f'(\phi_0) \phi_2 + f''(\phi_0) \phi_1^2/2!,$$

(6)

$$\phi_{3,t} = \phi_{3,xx} + f'(\phi_0) \phi_3 + f''(\phi_0) \phi_2 \phi_1 + f'''(\phi_0) \phi_1^3/3!,$$

(7)

etc. For (3), on taking $\phi_0 = 0$ we have $f'(\phi_0) = -a$, $f''(\phi_0) = 2(1 + a)$, $f'''(\phi_0) = -6$. We may also expand about $\phi_0 = 1$ but would then employ $\phi(x, 0) = 1$. In either case a recursive system of stochastic partial differential equations is obtained which can be solved in principle. All the equations are linear, have the same kernel and differ only in the nonhomogeneous term.

The Green function for (5) in the present problem may be written

$$G(x, y; t) = L^{-1} \exp(\gamma t) \left( 1 + 2 \sum_{n=1}^{\infty} \phi_n(x) \phi_n(y) \exp(-\mu_n t) \right), \quad t > 0,$$

(8)

where $\gamma = f'(\phi_0)$, $\phi_n(x) = \cos(n \pi x/L)$ and $\mu_n = n^2 \pi^2 / L^2$. The solution of (5) is then

$$\phi_1(x, t) = \int_0^L \int_0^t G(x, y; t-s) \left[ \alpha \, dy \, ds + \beta \, dW(y, s) \right],$$

(9)

where the second integral is a stochastic integral [13]. The expectation of $\phi_1$ is just $\alpha (e^{\gamma t} - 1)/\gamma$. Having $\phi_1$ we may integrate (6) and so on. Since

$$\phi_2(x, t) = \frac{\gamma}{2!} \int_0^L \int_0^t G(x, y; t-s) \phi_1^2(y, s) \, dy \, ds,$$

(10)

where $\gamma'' = f'''(\phi_0)$, we can find $E(\phi_2(x, t))$ from the variance of $\phi(x, t)$ which is itself easily calculated. Thus we get $E(\phi(x, t))$ to order $\epsilon^2$ for any $t$. We find

$$E(\phi(x, t)) \rightarrow \phi_0 - \frac{\epsilon \alpha}{\gamma} + \frac{\epsilon^2 \gamma'}{2} \left[ \frac{\beta^2}{2L} \left( \frac{1}{\gamma^2} - \frac{1}{\gamma} \sum_{n=1}^{\infty} \frac{1}{\mu_n - \gamma} + \sum_{n=1}^{\infty} \frac{\phi_{2n}(x)}{\mu_n - \gamma} \right) - \frac{\alpha^2}{\gamma^3} \right] + \mathcal{O}(\epsilon^2).$$

(11)

Choosing $\phi_0 = 0$, so the expansion is about background or resting level, we find that the noise elevates the mean value of $\phi(x, \infty)$ since $\gamma < 0$ and $\gamma'' > 0$.

The calculation of the covariance involves multiple stochastic integrals. We have
\[
\text{Cov}(\phi(x, s), \phi(y, t)) = \epsilon^2 \text{Cov}(\phi_1(x, s), \phi_1(y, t)) + \epsilon^3 \left[ \text{Cov}(\phi_1(x, s), \phi_2(y, t)) + \text{Cov}(\phi_2(x, s), \phi_1(y, t)) \right] + O(\epsilon^4). \tag{12}
\]

The \(O(\epsilon^2)\) term is easily obtained from previous results \([10, 12]\). The two \(O(\epsilon^3)\) terms require separate considerations. These contain eightfold integrals and products of stochastic integrals in the plane. For example, we find with \(s < t\),

\[
\text{Cov}(\phi_1(x, s), \phi_2(y, t)) = \beta^2 \alpha \gamma \left\{ \int_0^t \int_0^t G(x, x_1; s-s_1) \, dW(x_1, s_1) \right\} \times \left[ \int_0^t G(y, y_1; t-t_1) \left( \int_0^t G(y_1, y_2; t_1-t_2) \, dy_2 \, dt_2 \right) \left( \int_0^t G(y_1, y_3; t_1-t_3) \, dy_3 \, dt_3 \right) \right] \right\} 
\]

\[
= \frac{\beta^2 \alpha \gamma}{\gamma} \left[ \int_0^t (e^{\gamma t} - 1) \left( \int_0^t G(x, y; s+s_1) \, ds_1 \right) \, dt_1 
+ \int_0^t (e^{\gamma t} - 1) \, dt_1 \int_0^t G(x, y; s+s_1) \, ds_1 \right]. \tag{13}
\]

which is easily evaluated using (8). Similarly the other \(O(\epsilon^3)\) term in (12) can be found. The variance of \(\phi(x, t)\) at any \(x\) and any \(t\) is then known as is the correlation \(E(\phi(x, s)\phi(y, t))\) to order \(\epsilon^3\).

Putting \(t = s + \tau\), we let \(s \to \infty\) and find the limiting covariance of \(\phi(x, s)\) and \(\phi(y, s + \tau)\) as

\[
K(x, y; \tau) = \frac{\beta^2 \epsilon^2 e^{\frac{\tau}{2L}}}{2L} \left[ \left( \frac{\alpha \gamma e \tau}{\gamma} - 1 \right) \left( \frac{1}{\gamma} + 2 \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{(\gamma - \mu_n)^2} e^{-\mu_n \tau} \right) \right] + O(\epsilon^4). \tag{14}
\]

We may then find the spectral density at \(x\) which is the Fourier transform of \(K(x, x; \tau)\). This is, assuming \(\gamma < 0\),

\[
f_k(\omega) = \frac{\epsilon^2 \beta^2}{2\pi L} \left[ \frac{1}{\gamma^2 + \omega^2} + 2 \sum_{n=1}^{\infty} \frac{\phi_n^2(x)}{(|\gamma| + \mu_n)^2 + \omega^2} \right] + \frac{2\epsilon \alpha \gamma}{|\gamma|} \left( \frac{\gamma^2}{|\gamma|^2 + \omega^2} + 2 \sum_{n=1}^{\infty} \frac{\phi_n^2(x)(|\gamma| + \mu_n)}{((|\gamma| + \mu_n)^2 + \omega^2)^2} \right) + O(\epsilon^4). \tag{15}
\]

The \(O(\epsilon^3)\) terms are not present in the linear case and since \(\alpha\) can be arbitrarily large, these terms may even dominate the spectrum. The explicit time-dependent results will be useful in conjunction with computer simulations \([14]\).

In many previous models of fluctuating nerve membrane potential the waiting time for a threshold crossing has been employed to estimate the time between action potentials. This has been necessary because the systems employed were linear and had no natural threshold properties \([9, 12]\). Using nonlinear equations such as those of Hodgkin–Huxley or Fitzhugh–Nagumo obviates the determination of first passage times. A definite wave
of potential cannot exist with a uniform random forcing term but the moments of the potential may exhibit wave phenomena. Hence the present approach is an advance towards the solution of the problem of neuronal activity under random forcing. We also expect this approach to be useful in the analysis of membrane noise. We have succeeded in finding exact asymptotic expressions for the statistical properties of a potential satisfying a nonlinear diffusion-reaction equation with random forcing. Further details will be given elsewhere. We will also consider the corresponding multidimensional problem in a forthcoming publication.

References