Reprint Series
3 August 1979, Volume 205, pp. 493–495

SCIENCE

Solitons in a Reaction-Diffusion System

Henry C. Tuckwell

Solitons in a Reaction-Diffusion System

Abstract. *Solitary waves in reaction-diffusion systems usually annihilate on collision. A nonlinear system of reaction-diffusion equations has been constructed which has solitons: solitary waves whose interaction in a collision results in the emergence of two solitary waves identical to the colliding waves.*

Solitons may be defined as solitary waves which asymptotically maintain their shape and velocity after a collision with other solitary waves (*1*). They were demonstrated numerically (*2*) and analytically (*3*) for the Korteweg-deVries equation and have been found in a number of physical systems (*1*).

In this report evidence is presented that solitons may exist in a nonlinear system of reaction-diffusion equations. The system has solitary waves and when two solitary waves collide we find solitary waves identical to the original waves emerging from the collision. The methods employed consist of numerical integration of the reaction-diffusion equations.

Reaction-diffusion systems of equations arise in many models of biological

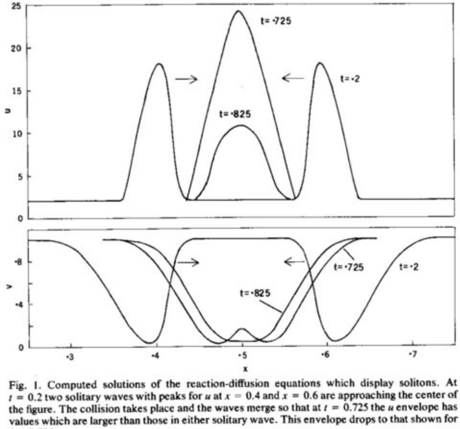

Fig. 1. Computed solutions of the reaction-diffusion equations which display solitons. At $t = 0.2$ two solitary waves with peaks for u at $x = 0.4$ and $x = 0.6$ are approaching the center of the figure. The collision takes place and the waves merge so that at $t = 0.725$ the u envelope has values which are larger than those in either solitary wave. This envelope drops to that shown for $t = 0.825$.

0036-8075/79/0803-0493$00.50/0

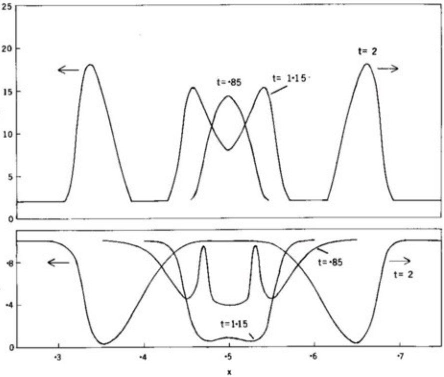

Fig. 2. The system has seen the modified reaction terms and at $t = 0.85$ both u and v have grown to achieve values sufficient to generate solitary waves. At $t = 1.15$ the postcollision waves have started to form, and by $t = 2$ they are fully developed with the same wave forms and speed as the incident colliding waves.

phenomena. Examples are the Hodgkin-Huxley (*4*) equations and the approximating equations of Fitzhugh (*5*) and Nagumo *et al.* (*6*) for propagation of the nerve impulse. When the solitary waves in these systems meet each other head on they annihilate one another.

Evidence for the existence of solitons was found by numerical computation in a two-component system of coupled nonlinear reaction-diffusion equations. Let the components be $u(x,t)$ and $v(x,t)$, where x and t are space and time variables. First, we consider the system

$$u_t = D_1 u_{xx} + F(u,v)$$
$$v_t = D_2 v_{xx} + G(u,v)$$

where the reaction terms are

$$F(u,v) = c_1 g(u)[V(u) - V_C(v)]\,[V(u) - V_K(u)] - c_2\{1 - \exp[-c_3(u - u_0)]\}$$

$$G(u,v) = c_4 g(u)[V(u) - V_C(v)] + c_5\{1 - \exp[-c_6(v_0 - v)]\}$$

with

$$g(u) = (1 + \tanh\{c_7[V(u) + V_T]\})H(u - u^*)$$

and

$$V(u) = 58\log_{10}\left(\frac{u + c_8}{c_9}\right)$$

$$V_K(u) = 58\log_{10}\left(\frac{u}{c_{10}}\right)$$

$$V_C(v) = 29\log_{10}\left(\frac{v}{c_{11} - kv}\right)$$

These equations are based on the evolution equations for potassium ion concentration, $u(x,t)$, and calcium ion concentration, $v(x,t)$, in the extracellular space of brain structures (*7*). The c_i, $i = 1, \ldots, 11$, and k are constants; $g(u)$ describes the calcium conductance of presynaptic membrane; $V(u)$ is the membrane potential; $V_K(u)$ is the potassium equilibrium potential; $V_C(v)$ is the calcium equilibrium potential; H is the Heaviside unit step function; and D_1 and D_2 are the diffusion coefficients for potassium and calcium ions. The subscripts x and t denote partial differentiation with respect to these variables.

The system of equations above predicts the qualitative and to some extent the quantitative behavior of potassium and calcium ion concentrations during cortical spreading depression (*8*). A local elevation of potassium ion concentration as an initial condition

$$u(x,0) = u_0 + 8\exp\left[-\left(\frac{x - 0.5}{0.025}\right)^2\right]$$

with u_0, the resting level of potassium, at 2 mM, and the calcium at resting level

$$v(x,0) = v_0$$

with $v_0 = 1$ mM, gives rise, with appropriate values of the constants ($c_1 = -3$, $c_2 = 208$, $c_3 = 10$, $c_4 = 0.3$, $c_5 = 2.08$, $c_6 = 10$, $c_7 = 0.11$, $c_8 = 9$, $c_9 = 180$, $c_{10} = 140$, $c_{11} = 0.05$, $k = 0.25$, $V_T = 45$, $u^* = 2.2$, $D_1 = 0.005$, and $D_2 = 0.00125$), to solitary waves of an increase in u and a decrease in v moving outward from the applied stimulus. When two waves propagate from remote stimuli and collide, they annihilate one another and u and v return to their resting values of u_0 and v_0.

At a fixed spatial point, x, a trajectory in the (u,v) plane can be traced as the solitary wave passes x. When two solitary waves collide one obtains a family of trajectories in the (u,v) plane at various spatial points. In particular, the collision trajectories at and near the center of the collision are quite removed from the solitary wave trajectory. The solitary wave trajectory and collision trajectories were found for the reaction-diffusion system above by numerical solution of the equations.

One can change $F(u,v)$ and $G(u,v)$ at values of (u,v) that do not arise on the solitary wave trajectory, and a solitary wave will still propagate. The idea is now to leave $F(u,v)$ and $G(u,v)$ the same at (and near, because the computations are numerical) the solitary wave trajectory but alter the reaction terms at (u,v) values that arise only on the collision trajectories. For examples, for the system above at $x = 0.5$ (the collision center) the values $(u,v) = (10.7, 0.0338)$ arise during the collision interaction. The values of $F(10.7, 0.0338)$ and $G(10.7, 0.0338)$ are such that u continues to decrease toward $u_0 = 2$ and v increases toward $v_0 = 1$. A change in the reaction terms in a small rectangle in the (u,v) plane containing the point (10.7, 0.0338) ensures that when these values of u and v arise in the collision, instead of u decreasing, an increase occurs. It was found after a few such changes were made that the result of a collision of two solitary waves was not a return to resting values but the emergence of two solitary waves of the same amplitude and velocity as the colliding waves. The modified system which has these apparent soliton solutions can be written

$$u_t = D_1 u_{xx} + F(u,v) + \sum_i a_i I_{A_i}(u,v)$$

$$v_t = D_2 v_{xx} + G(u,v) + \sum_i b_i I_{A_i}(u,v)$$

where I_{A_i} is the indicator function of the set $A_i = \{(u,v) | u \in (u_i^1, u_i^2),\ v \in (v_i^1, v_i^2)\}$, which takes the value 1 if $(u,v) \in A_i$ and is zero otherwise. The quantities a_i and b_i are constants and the little rectangles A_i do not contain points on the solitary wave trajectory.

Results are shown in Figs. 1 and 2. Solutions of the reaction-diffusion systems were computed by using Lees' modification of the Crank-Nicolson numerical procedure (*9*). Figure 1 shows the computed solitary waves of u and v traveling toward the center of the figure and the initial merging of the waves. In the unmodified system these collision envelopes collapse back to resting values. Figure 2 shows further interactions and the emergence of solitary waves after the collision in the modified system. To check that the modified system still gave solitary waves in response to an initial stimulus of locally elevated u, solutions of the modified equations were computed with $u(x,0)$ and $v(x,0)$ as given above. The result was two solitary waves traveling to the left and right, as had been the case for the original system. It is noteworthy that when slightly asymmetric initial data were employed for the modified system, when two solitary waves collided only one emerged from the collision.

The ramifications of the existence of solitons in reaction-diffusion systems are far-reaching. Models for the activity of populations of neurons have hinted at their existence (*10*) and they may be important in particle physics. The idea of solitons in neuroanatomic structures may be important in possible theories of memory. The reaction-diffusion system in which solitons have been found by numerical computation (which can give only evidence rather than proof of their existence) has been constructed from a system that arises in describing the evolution of ion concentrations in cortical structures. It is hoped that systems will be found whose reaction terms arise naturally and which give rise to soliton solutions. The aim of this investigation has been to obtain evidence that reaction-diffusion systems can support soliton solutions, which had not previously been suspected.

HENRY C. TUCKWELL*

Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada V6T 1W5

References and Notes

1. A. C. Scott, F. Y. F. Chu, D. W. McLaughlin, *Proc. IEEE* **61**, 1443 (1973).
2. N. J. Zabusky and M. D. Kruskal, *Phys. Rev. Lett.* **15**, 240 (1965).
3. N. J. Zabusky, *Phys. Rev.* **168**, 124 (1968).
4. A. L. Hodgkin and A. F. Huxley, *J. Physiol. (London)* **117**, 500 (1952).
5. R. Fitzhugh, *Biophys. J.* **1**, 445 (1961).
6. J. Nagumo, S. Arimoto, S. Yoshizawa, *Proc. IRE* **50**, 2061 (1962).
7. H. C. Tuckwell and R. M. Miura, *Biophys. J.* **23**, 257 (1978).
8. C. Nicholson, G. Ten Bruggencate, R. Steinberg, H. Stockle, *Proc. Natl. Acad. Sci. U.S.A.* **74**, 1287 (1977); R. P. Kraig and C. Nicholson, *Neuroscience* **3**, 1045 (1978).
9. M. Lees, in *Numerical Solution of Partial Differential Equations*, W. F. Ames, Ed. (Barnes & Noble, New York, 1969), p. 193.
10. R. L. Beurle, *Philos. Trans. R. Soc. London Ser. B* **240**, 55 (1956).
11. D. Cope kindly provided a computer program and useful discussion. Supported in part by National Research Council of Canada grants A 4559 and A 9259.
* Address from September 1979: Department of Biomathematics, School of Medicine, University of California, Los Angeles 90024.

30 April 1979