

## DETERMINATION OF THE ELECTRICAL POTENTIAL OVER DENDRITIC TREES BY MAPPING ONTO A NERVE CYLINDER

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(Received March 4, 1984)

### Abstract

A mapping from a dendritic tree to a nerve cylinder is described. The *effective potential* is defined as a weighted sum of the depolarizations at all points which are the same electrotonic distance from the origin. When the three halves power law is obeyed at branch points, the effective potential satisfies a single cable equation. Under mild assumptions, the solution of this cable equation yields the potential at the soma, and, when there is only one dendritic tree, on the trunk. The potential over the remainder of the tree can be found by solving a sequence of cable equations with two-point boundary conditions. The effects of input location and branching on the response at the soma with reference to the analysis of postsynaptic potentials are considered. When the three-halves power law for diameters is obeyed and when all dendritic terminals are the same electrotonic distance from the soma with the same type of boundary condition at each terminal, it is shown that the somatic potential depends only on the input current strength and distance from input to soma; that is, the response is independent of the pattern of branching. A numerical example is provided in which the synaptic input is asymmetric and the effects of timing of the excitation relative to the inhibition are found. In a further example, for an asymmetric tree with two orders of branching, exact expressions are found for the depolarization over the entire dendritic tree. A proof of the main result is contained in the appendix.

### 1. Introduction

Under certain conditions, the subthreshold depolarization (membrane potential minus resting potential) of a single nerve cylinder is well-approximated by solutions of the cable equation

$$(1.1) \quad c_m r_m V_t = \frac{r_m}{r_i} V_{xx} - V + r_m I, \quad 0 < x < L, \quad t > 0,$$

with appropriate boundary conditions. Here  $V=(x,t)$  is the depolarization in volts,  $I=I(x,t)$  is the current density in amperes  $\text{cm}^{-1}$ ,  $x$  is distance in cm,  $t$  is time in sec and subscripts  $x$  and  $t$  indicate partial derivatives. The constants  $c_m, r_m, r_i$  and  $L$  are, respectively, the membrane capacitance per unit length ( $F \text{ cm}^{-1}$ ), the resistance of a unit length of membrane times unit length ( $\Omega \text{ cm}$ ), the axial resistance per unit length ( $\Omega \text{ cm}^{-1}$ ), and the length of the cylinder (cm).

Hodgkin and Rushton (1946) showed that equation (1.1) was applicable to certain axons provided that the stimulating current was less than about half that required to induce action potentials. It has often been assumed since that it is also applicable to the dendrites of certain nerve cells (Rall, 1959, and subsequent papers; more recently, Rall and Rinzel, 1973 and Horwitz, 1981. The latter two papers contain extensive references to earlier work.)

Although equation (1.1) omits the effects of synaptic reversal potentials (MacGregor, 1968), the practical advantage of using it to approximate the behavior of a nerve cylinder is that its solutions may be obtained with little difficulty for most relevant boundary conditions. However, when one considers branching structures such as dendritic trees or telodendria, the situation becomes much more complex. One is forced to include boundary conditions at each branch point and each terminal. This leads to very complicated expressions for even the Laplace transform of the solution.

Rall (1962) showed that in cases where there is enough symmetry, it is possible to find the potential on the entire tree, using his idea of the equivalent cylinder. The method was, first to change variables to electrotonic distance (which we shall describe below) and then to map the tree onto a single cylinder. After solving for the potential on the cylinder, he could recover the potential on the tree by using symmetry considerations.

The symmetry requirements for the equivalent cylinder procedure are:

- (a) all dendritic terminals are the same electrotonic distance from the origin;
- (b) the boundary conditions at all the terminals are the same;
- (c) at each branch point the three-halves power of the diameter of the parent cylinder equals the sum of the three-halves powers of the diameters of the daughter cylinders;
- (d) the input current density divided by the three-halves power of the diameter is the same at all points which are the same electrotonic distance from the origin.

If one is only interested in current injection at the soma, then condition (d) is satisfied. Otherwise, for example in the case of synaptic input on the dendrites, condition (d) is a serious restriction and is usually violated. It has been pointed out (Jack, Noble and Tsien, 1975) that a mapping from tree to cylinder may still be useful and we verify and elaborate on this. Our main concern is to demonstrate that the potential at the soma, and on the trunk if there is a single dendritic tree, may be found by solving a single cable equation if conditions (b) and (c) hold and the boundary conditions at each dendritic terminal are of the same type.

Rall and Rinzel (1973) and Rinzel and Rall (1974) were mainly concerned with a class of model neurons in which there were several identical binary dendritic trees with symmetric branching. They used a clever adaptation of the method of images

and obtained an expression for the time-dependent potential in response to current injection at a single branch terminal when the three-halves power law was obeyed and when all dendritic terminals were the same electrotonic distance from the origin. In the Appendix to their (1974) paper they obtained the Laplace transform of the voltage when the input current was not restricted to a terminal and when the assumptions of equal trunk diameters and equal terminal distances were relaxed.

Our approach treats roughly the same class of trees but is simpler and more direct. We consider a class of trees satisfying (a) and (c) but do not require (b) and (d). The key to the method is a map of the potentials on the various branches into a certain weighted sum, called the *effective potential*, which itself satisfies the cable equation. The effective potential equals the true potential on the trunk if there is one dendritic tree and it equals the true potential at the soma if there are several dendritic trees. One may thus obtain the trunk (soma) potential by solving a single cable equation with boundary conditions. The method also applies when the soma is represented as a lumped R-C circuit.

As in Rall's case, the present method involves reducing the tree to a single cylinder. It is not, however, an equivalent cylinder — that is, the potential on the tree determines the potential on the cylinder but the potential on the cylinder does not determine the potential throughout the tree. That could be done in Rall's case because (a)–(d) imply that the potential is the same in all branches at points which are the same electrotonic distance from the origin. This is not true for an arbitrary input or when the boundary conditions at all terminals are not the same. However, we show it is still possible to recover the potential throughout the model neuron recursively, by solving a sequence of cable equations. It is emphasized that our results are contingent of the validity of the three-halves power law and that many of them can be obtained by other methods (c.f., Rinzel and Rall, 1974).

## 2. A Change of Variables

Consider a dendritic tree. Let the left-hand endpoint of the tree (the soma) be at the origin and let  $x$  be the distance from the origin to a given point on the tree as measured along the branches. If  $x$  is not a branch point, the potential will satisfy (1.1) in a neighborhood of  $x$ .

Some of the constants in (1.1) depend on the diameter of the cylinder, so let us make this dependence explicit by rewriting the equation. Let  $\delta$  be the membrane thickness,  $C_m$  the membrane capacitance per unit area,  $\rho_i$  and  $\rho_m$  the resistivities of the intracellular medium and membrane, respectively. We assume that these are all constants of the neuron; that is, they are the same for all dendrites. Let us recall the change of variables introduced by Rall for his equivalent cylinder. Define dimensionless variables  $X$  and  $T$  locally by

$$(2.1) \quad X = x/\lambda, \quad T = t/\tau,$$

where  $\lambda = (r_m/r_i)^{1/2} = (\rho_m \delta d / 2\rho_i)^{1/2}$  is the characteristic length and

$\tau = c_m r_m = \rho_m \delta C_m$  is the time constant. Note that  $\lambda$  varies with the dendrite, since it is proportional to  $\sqrt{d}$ , while  $\tau$  is a constant of the neuron. Define

$$(2.2) \quad k = \frac{2}{\pi C_m} \left( \frac{\rho_i}{\rho_m \delta} \right)^{1/2}$$

which is a constant of the neuron (with value about  $5 \times 10^4 F^{-1} \text{cm}^{3/2}$  for cat spinal motoneurons) and let

$$(2.3) \quad \bar{I}(X, T) = \lambda \tau I(x, t).$$

Then  $\bar{I}$  is the current density in coulombs (i.e., coulombs per unit dimensionless time per unit dimensionless distance), and the cable equation (1.1) becomes

$$(2.4) \quad V_T = V_{XX} - V + kd^{-3/2} \bar{I},$$

where  $d$  is the diameter in cm. Note that although we work with the dimensionless space variable  $X$ , diameters will be in cm throughout since this preserves the familiar "three-halves-power law".

We can now define the *electrotonic distance* of a point on the dendritic tree from the origin: it is the distance from the soma measured along the branches, where on each branch we measure the distance in characteristic lengths for that branch. The electrotonic distance from the origin is hereafter denoted by  $x$  and the word "distance" will mean electrotonic distance.

### 3. The Main Results

We consider a dendritic tree either like that in Figure 1A or like that in Figure 1B. We shall suppose throughout this section that distance is measured in electrotonic length and time is in time constants (so that the change of variables of Section 2 has already been made). We will use  $x$  and  $t$ , for these variables respectively.

Let the left-hand end point (soma) be at the origin. The right-hand extremities (dendritic terminals) are labeled  $T_1, \dots, T_n$ . Branching occurs at distances  $0 = x_0 < x_1 < \dots < x_m$  from the origin. As we have indicated, there may be two or more daughter cylinders at a given branch point.

We require the *three-halves-power law*: if the diameter of the parent cylinder is  $D_0$  and those of the daughter cylinders are  $D_1, D_2, \dots$ , then

$$(3.1) \quad D_0^{3/2} = D_1^{3/2} + D_2^{3/2} + \dots$$

Let the electrotonic distances of the dendritic terminals from the origin be  $L_1, \dots, L_n$ . Suppose there exist  $n_i$  cylinders between  $x_i$  and  $x_{i+1}$ . Let their corresponding depolarizations, current densities and diameters be  $V_{ij}(x, t)$ ,  $I_{ij}(x, t)$  and  $d_{ij}$ , respectively, where  $j = 1, 2, \dots, n_i$ ,  $x_i \leq x \leq x_{i+1}$ ,  $t > 0$ , and  $i = 0, 1, \dots, m$ .

It follows from (3.1) that the sum  $d_{i1}^{3/2} + d_{i2}^{3/2} + \dots$  is the same for each  $i$ . Thus define

$$(3.2) \quad d_0 = \left( \sum_{j=1}^{n_i} d_{ij}^{3/2} \right)^{2/3}.$$

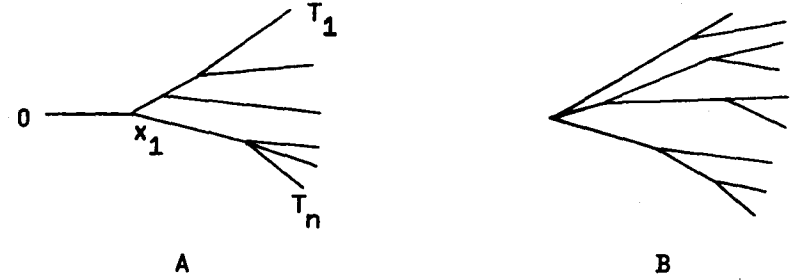


Fig. 1. Schematic neurons considered in Theorem 1 with, (A), one and (B), several dendritic trees emanating from the origin or soma.

Notice that if the neuron has a single trunk, then  $d_0$  is its diameter.

We now map the dendritic tree onto the line segment  $[0, \max L_n]$  by mapping each point of the tree onto its electrotonic distance from the origin. We call this line segment the *effective cylinder*. We then map the potentials on the dendrites onto the *effective potential*  $V$ :

$$(3.3) \quad V(x, t) = \sum_{j=1}^{n_i} \left( \frac{d_{ij}}{d_0} \right)^{3/2} V_{ij}(x, t), \quad x_i \leq x \leq x_{i+1}, \quad i = 0, 1, \dots, m.$$

We assume that the potential on each segment satisfies (2.4), where  $d$  is the diameter (in cm) of that segment. We assume, moreover, that at each branch point the potential is continuous and that the longitudinal current is conserved.

Notice that  $V$  has a different formula on each interval  $(x_i, x_{i+1})$ . Nevertheless, as long as  $x < \min(L_j)$ ,  $V$  is continuous. Indeed, it is continuous in the interior of each segment, and it is continuous at a branch point because the potential itself is continuous there, while (3.2) implies that  $\sum_{j=1}^{n_i} (d_{ij}/d_0)^{3/2} = 1$  for each  $j$ . In fact we have the following result, which is proved in the Appendix.

#### Theorem

Let  $L$  be the minimum of  $L_1, L_2, \dots, L_n$ . Then the effective potential  $V$  satisfies the cable equation

$$(3.4) \quad V_t = V_{xx} - V + kd_0^{-3/2} I, \quad 0 < x < L, \quad t > 0,$$

where

$$(3.5) \quad I(x, t) = \sum_{j=1}^{n_i} I_{ij}(x, t), \quad x_i < x < x_{i+1}, \quad i = 0, 1, \dots, m.$$

This theorem will be proved in the appendix. As it is, the theorem is quite general — the only real restriction on the neuron is that its branches satisfy the three-halves-power law (3.1). The cable equation (3.4) applies at those distances which are less than the distance from the origin to the closest dendritic terminal. To apply the theorem, however, we need to impose boundary conditions, and this requires a further

restriction. Henceforth all dendritic terminals will be assumed to be at the same distance from the origin. We first take the case of a single tree, in which case the potential at the soma and trunk is  $V_{01}$ .

### Corollary 1

Assume that  $V_{01}$  satisfies a given boundary condition at  $x=0$  and that all terminals are at the same distance from the soma ( $L_1=L_2=\dots=L_n=L$ ). Assume there are given boundary conditions at the terminals

$$(3.6) \quad \alpha V_{mj}(L,t) + \beta V_{mj,x}(L,t) = \gamma_j(t), \quad j=1, \dots, n,$$

where  $\alpha$  and  $\beta$  are constants. Assume further that the  $V_{ij}$  satisfy the initial conditions

$$(3.7) \quad V_{ij}(x,0) = v_{ij}(x), \quad x_i < x < x_{i+1}, \quad i=0, 1, \dots, m; \quad j=1, \dots, n_i.$$

Then the effective potential  $V$  satisfies the cable equation (3.4) on  $0 < x < L, t > 0$ , along with the given boundary condition at  $x=0$ , the boundary condition at  $x=L$ :

$$(3.8) \quad \alpha V(L,t) + \beta V_x(L,t) = \sum_{j=1}^n \left( \frac{d_{mj}}{d_0} \right)^{3/2} \gamma_j(t),$$

and the initial condition:

$$(3.9) \quad V(x,0) = \sum_{j=1}^{n_i} \left( \frac{d_{ij}}{d_0} \right)^{3/2} v_{ij}(x), \quad x_i < x < x_{i+1}, \quad i=0, \dots, m.$$

Indeed,  $V$  satisfies the cable equation by the above theorem, it satisfies the boundary condition at  $x=0$  since  $V=V_{01}$  in a neighbourhood of the origin, and (3.8) and (3.9) follow by linearity.

Now consider the case in which there are a number,  $M$ , of dendritic trees emanating from a common soma.

### Corollary 2.

Assume that all dendritic terminals are the same electrotonic distance  $L$  from the origin, and assume that the net current entering the origin is  $I_0(t)$ . Suppose that the boundary conditions (3.6) and the initial conditions (3.7) hold. Then the effective potential  $V$  satisfies the cable equation (3.4) on  $0 < x < L, t > 0$ , along with the initial conditions (3.9), boundary conditions (3.8) at  $x=L$  and, at  $x=0$ ,

$$(3.10) \quad I_0 + \kappa V_x(0,t) = 0,$$

where  $\kappa$  is a constant given by

$$(3.11) \quad \kappa = \frac{\pi \tau}{2} \left( \frac{d_0^3}{\delta \rho_i \rho_m} \right)^{1/2}$$

and where  $I_0$  is in coulombs.

To see this, note first that, except at  $x=0$ , the situation is exactly the same as in Corollary 1, so the only thing that needs to be justified is (3.10). But by Kirchoff's current law the sum of the input current  $I_0$  and the longitudinal currents at the soma in the trunks of the dendritic trees must be zero. Hence, with dimensionless space and time variables,

$$(3.12) \quad I_0 + \tau \sum_{j=1}^{n_0} \frac{1}{\bar{r}_{i,0j}} V_{0j,x} = 0,$$

where  $\bar{r}_{i,0j}$  is the internal resistance of a characteristic length of the  $j$ -th trunk. The quantities  $\bar{r}_{i,0j}$  are inversely proportional to the diameters to the three-halves power,  $d_{0j}^{3/2}$ , and on evaluating the physical constants we obtain (3.10) and (3.11).

We have given the boundary condition at  $x=0$  in terms of the current since this makes it easy to derive the boundary conditions at the origin in cases of physiological interest. For instance, if there is no current "injected" at the soma, (3.10) becomes

$$(3.13) \quad V_x(0,t) = 0.$$

Another situation often considered is the attachment of a "lumped soma" at the origin, in which a resistance  $R_s$  and a capacitance  $C_s$  in parallel represent the soma. (See Fig. 2). Then, assuming the time constant of the soma is the same as that of the dendrites and that a current  $I_s(t)$  is injected at the soma, the boundary condition at  $x=0$  is

$$(3.14) \quad I_s(t) - R_s^{-1} [V_i(0,t) + V(0,t)] + \kappa V_x(0,t) = 0.$$

Notice that when Corollary 2 is applied, the effective potential  $V$  at  $x=0$  is the potential at the soma by virtue of (3.2).

One boundary condition which is not easy to read of from (3.10) is the case of a voltage clamp,  $V_c$ , at the soma. However, we then have directly

$$(3.15) \quad V(0,t) = V_c.$$

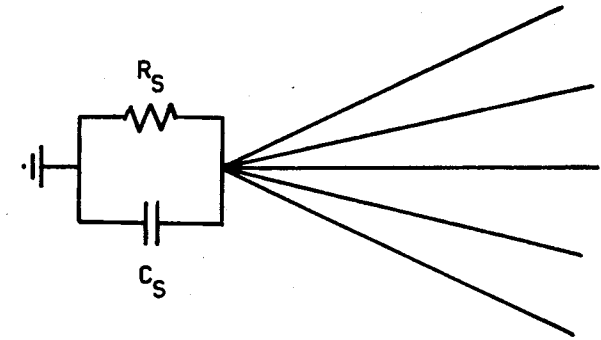


Fig. 2. Lumped soma termination at the origin of several dendritic trees, each tree being represented by its effective cylinder.

#### 4. Applications of Corollary 1

For a neuron with a single dendritic tree, Corollary 1 enables one to find the depolarization at the soma and on the trunk in the case of an arbitrary configuration of input currents with the same ease that the depolarization on a single nerve cylinder can be found. The reason for this is that the mapping (3.3) from tree to cylinder is an identity on the trunk:

$$(4.1) \quad V(x,t) = V_{01}(x,t), \quad 0 \leq x \leq x_1,$$

and  $V(x,t)$  is found by solving a single cable equation with two-point boundary conditions (i.e., at  $x=0$  and  $x=L$ ). This is useful because one is primarily interested in the response at and near the soma, since this is the presumed site of a recording electrode and is possibly the neuron's trigger zone. The depolarization can be found at points not on the trunk, but more cable equations, again with two-point boundary conditions, need to be solved — see below.

##### *Effect of location of an input on the dendritic tree*

Suppose an input current  $i(t)$  arises on a single segment of the dendritic tree at a point which is an electrotonic distance  $y$  from the origin. Suppose further that the conditions for Corollary 1 are fulfilled. Then the depolarization on the trunk satisfies

$$(4.2) \quad V_t = -V + V_{xx} + kd_0^{-3/2} \delta(x-y)i(t), \quad 0 < x < L, \quad t < 0,$$

restricted to  $0 \leq x \leq x_1$ . But equation (4.2) is precisely the equation obtained if the trunk is extended from  $x_1$  to  $L$  and the input current is applied on the extended cylinder at the same distance  $y$  from the origin. We therefore have the following principle.

*The depolarization at the soma and on the trunk in response to a given input current occurring at any point on the dendritic tree depends only on the electrotonic distance of the input from the soma and not on the geometrical details of the dendritic tree.*

Contained in this statement is the fact that the time course of the response at the soma and on the trunk is, for a given input current, independent of the geometrical details of the tree and depends only on  $y$ . This conclusion is based on the validity of the three assumptions (a) the three-halves power law holds at branch points, (b) all dendritic terminals are the same distance from the origin, and (c) the boundary conditions at each dendritic terminal are of the same type as indicated in Equ. (3.6). The question naturally arises as to how robust this result is when the three-halves power law is not obeyed, because although some neurons, e.g. cat spinal motoneurons, roughly follow the three-halves law, others do not. For example, cerebellar Purkinje cells have the sum of *squares* of the diameters of daughter cylinders approximately equal to the square of the diameter of the parent cylinder (Hillman, 1979). Thus for Purkinje cells, with two daughter cylinders,  $D_0^2 = D_1^2 + D_2^2$ ,

and assuming  $D_1 = D_2$  we have  $D_1 \approx 0.71 D_0$  compared with  $D_1 \approx 0.63 D_0$  for the three-halves law case. Calculations of Horwitz (1981, 1983) for a synaptic input applied to one daughter cylinder of a tree with one branch point indicate only a small difference in the potential at the origin in these cases. We expect therefore that the above conclusion will be quite robust with regard to departures from the three-halves law.

##### *Response to an arbitrary input*

Given input current densities  $I_{ij}$ ,  $i=0,1,\dots,m$ ;  $j=1,2,\dots,n_i$ , on the various segments of the dendritic tree, the cable equation satisfied by the effective potential is given by (3.4) and (3.5). Suppose that with suitable given boundary conditions, the Green's function (i.e., impulse response) is  $G(x,y;t)$ . Then the solution of (3.4) with zero initial data is given by the formula

$$(4.3) \quad V(x,t) = \int_0^L \int_0^t G(x,y;t-s) I(y,s) ds dy,$$

thus enabling the depolarization on the trunk to be found for an arbitrary input configuration.

##### *Examples*

###### (i) *Response to an arbitrary space-time pattern of impulse currents*

The Green's function  $G(x,y;t)$  is the solution of (3.4), with given boundary conditions, when a unit charge (1 coulomb) is delivered at  $t=0$  at the point  $y$ . Consider now a dendritic tree fulfilling the conditions of the Theorem and Corollary 1 with the same given boundary conditions. Suppose charges  $q_{ij}$  are delivered at times  $t_{ij}$  at the points  $y_{ij}$  on segment  $ij$ . Then the effective potential  $V$  satisfies the equation

$$(4.4) \quad V_t = -V + V_{xx} + kd_0^{-3/2} \sum_{i=0}^m \sum_{j=1}^{n_i} q_{ij} \delta(x-y_{ij}) \delta(t-t_{ij}).$$

It follows that the depolarization on the trunk is

$$(4.5) \quad V(x,t) = \sum_{i=0}^m \sum_{j=1}^{n_i} q_{ij} G(x,y_{ij};t-t_{ij}),$$

restricted to  $0 \leq x \leq x_1$ .

###### (ii) *Response to uniform stimulation*

Consider a dendritic tree satisfying the three-halves-power law, and let us ask what the response is if a constant current with uniform density *per unit area* is applied over the surface of the dendritic tree, where we measure area in the original units, not in

terms of characteristic lengths. Suppose this uniform current density is  $i_A$  amps  $\text{cm}^{-2}$ , corresponding to  $i_A \pi d$  amps  $\text{cm}^{-1}$  on a cylinder of diameter  $d$  cm. On changing to dimensionless space and time variables, the cable equation for a single cylinder becomes

$$(4.6) \quad V_t = -V + V_{xx} + k_1 i_A,$$

where  $k_1 = \pi k \lambda \tau / \sqrt{d}$ . If each segment of the dendritic tree receives the same constant current density per unit area, then on segment  $ij$  the forcing term is  $k_1 i_A d_{ij}^{3/2} / k$ . Thus the effective potential satisfies the equation

$$(4.7) \quad V_t = -V + V_{xx} + I,$$

where

$$(4.8) \quad I(x, t) = k_1 i_A \sum_{j=1}^{n_i} (d_{ij} / d_{01})^{3/2}, \quad x_i \leq x \leq x_{i+1}, \quad i = 0, 1, \dots, m,$$

However, the three-halves-power law implies that

$$\sum_{j=1}^{n_i} (d_{ij} / d_{01})^{3/2} = 1, \quad i = 0, 1, \dots, m,$$

so that the cable equation is

$$(4.9) \quad V_t = -V + V_{xx} + k_1 i_A,$$

which corresponds to an extension of the trunk with uniform stimulation.

It is seen from this example and the previous discussion that as far as the effect of stimuli at the soma and trunk are concerned, the neuron loses nothing by attenuation due to branching. No input is discriminated against by virtue of its position on the dendritic tree except with regard to electrotonic distance from the origin. Branching which satisfies the three-halves-power law does not serve to attenuate but only to create the possibility of a greater number of inputs (synapses). This enables the neuron to achieve the integration and discrimination of complex input patterns and still remain reasonably compact. By virtue of branching, more inputs are possible at a given electrotonic distance from the soma. If the inputs are from a common source, then branching can serve to increase the response at the soma and trunk. For example, the parallel fiber input to the dendritic trees of cerebellar Purkinje cells is believed to make multiple connections with several dendritic segments at about the same electrotonic distance from the soma, leading to a greatly enhanced postsynaptic potential. Damaged or lost dendritic segments would lead to loss of input response in a proportional fashion.

### (iii) Current injection at a single terminal

Consider a neuronal dendritic tree satisfying the conditions of the Theorem and Corollary 1. Suppose that current  $i(t)$  is injected into the  $j$ -th terminal. The boundary

condition there is

$$(4.10) \quad \frac{1}{\tau r_{mj}} V_{mj,x}(L, t) = i(t),$$

where  $r_{mj}$  is the internal resistance of a characteristic length ( $\lambda_{mj}$ ) of the dendrite whose terminal receives the input, and  $i(t)$  is in coulombs per unit dimensionless time. By Corollary 1, the depolarization at the soma and on the trunk is the solution of

$$(4.11) \quad V_t = -V + V_{xx}, \quad 0 < x < L, \quad t > 0,$$

restricted to  $0 \leq x \leq x_1$ , with a given boundary condition at  $x=0$  and, from (3.8),

$$(4.12) \quad \frac{1}{\tau r_{mj}} V_x(L, t) = (d_{mj} / d_{01})^{3/2} i(t)$$

The constants in this last equation can be rearranged to give

$$(4.13) \quad \frac{1}{\tau r_{01}} V_x(L, t) = i(t),$$

where  $r_{01}$  is the internal resistance of a characteristic length ( $\lambda_{01}$ ) of the trunk. Thus again, with current injection at a single terminal, the dendritic tree, insofar as the voltage at the soma and on the trunk is concerned, acts as an extension of the trunk. The branching, no matter how profuse, has no effect on the propagation of voltage from terminals to the trunk and soma, provided that the three-halves power law and assumptions we have made concerning the dendritic terminals are valid.

### (iv) A numerical example

To demonstrate the use of Corollary 1, consider the schematic dendritic tree of Figure 3. This shows afferent fibers coursing through the distal dendrites and making several excitatory synaptic contacts. The synapses occur irregularly over the dendrites, rendering the symmetry requirements of the equivalent cylinder invalid. These are at  $y_1, y_2, \dots, y_n$  from the origin. For simplicity we suppose there is just one inhibitory synapse on the trunk at a distance  $y$  from the soma. We assume that a volley in the excitatory afferents leads to impulse currents in the postsynaptic cell at the synapses, and similarly for the inhibitory synapse. Suppose the time of arrival of the excitation is  $t_1$  and that of the inhibition is  $t_2$ . Then, providing the dendritic tree satisfies the conditions of Corollary 1, if the same charge is delivered at each synapse, the depolarization at the soma is a multiple of

$$(4.14) \quad V(t) = \left[ \sum_{i=1}^n G(0, y_i; t - t_1) \right] - G(0, y; t - t_2).$$

Values of  $V(t)$  were computed for a dendritic tree with  $L=2$ ,  $n=8$ ,  $y_i = 1.2 + i/10$ ,  $i=0, 1, \dots, 7$ ,  $y=.3$  and with sealed end conditions at  $x=0$  and  $x=L$ . The Green's function for these boundary conditions has the following two representations:

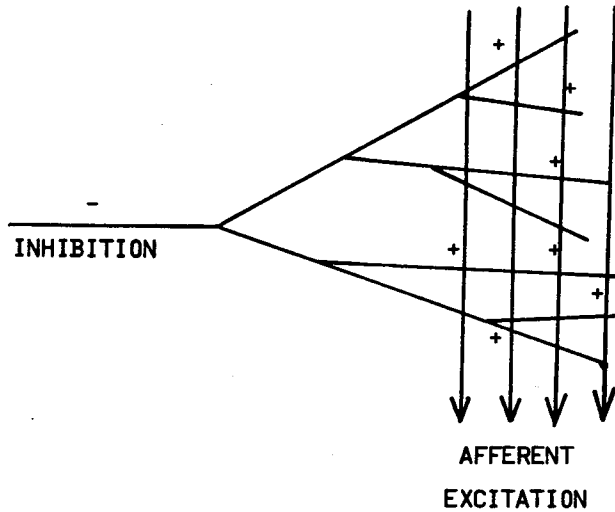


Fig. 3. Schematic neuron for numerical example. Afferent fibers deliver excitation at synapses located on distal dendrites and inhibition occurs on the trunk.

(4.15)

$$G(x,y;t) = \frac{kH(t)e^{-t}}{d^{3/2}L} \left[ 1 + 2 \sum_{n=1}^{\infty} \cos(n\pi x/L) \cos(n\pi y/L) \exp\{-n^2\pi^2 t/L^2\} \right],$$

(4.16)

$$G(x,y;t) = \frac{kH(t)e^{-t}}{d^{3/2}} \sum_{n=-\infty}^{\infty} \left[ \exp\left\{-\frac{(x-2nL-y)^2}{4t}\right\} + \exp\left\{-\frac{(x-2nL+y)^2}{4t}\right\} \right],$$

where  $H(\cdot)$  is the unit step function

$$(4.17) \quad H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

The series (4.15) converges rapidly for large values of  $t$ , whereas (4.16) converges rapidly for small  $t$ .

In the calculations we considered the effects of the timing of the excitation relative to the inhibition. Thus we let  $t_1 = t_2$ ,  $t_1 = t_2 + .5$  and  $t_1 = t_2 - .5$  representing excitation at the same time as, later than, and earlier than the inhibition. The results are shown in Figure 4. It can be seen that although the shapes of the complex postsynaptic potentials are quite different in these three cases, their peak amplitudes do not differ very much. In contrast, calculations including synaptic reversal potentials have shown that inhibition may effectively block excitation at *neighboring* synapses when inhibitory and excitatory conductance changes overlap in time (Koch, Poggio and Torre, 1983).

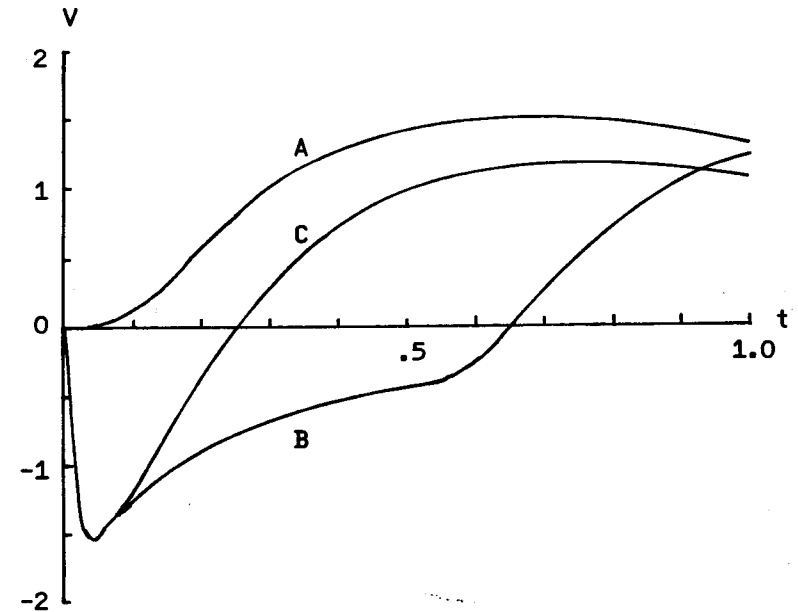


Fig. 4. Computed response at the soma when the excitation arrives before (A), after (B), and at the same time (C), as the inhibition in the numerical example.

## 5. Finding the Depolarization Over the Entire Dendritic Tree

Corollary 1 enables the depolarization at the soma and on the trunk to be found immediately by solving a cable equation with two-point boundary conditions for an arbitrary configuration of input currents. The depolarization over the remainder of the tree can be found by solving a sequence of cable equations with two-point boundary conditions. The procedure, in outline, is as follows.

We first collapse the whole tree into a single cylinder and solve the corresponding cable equation to obtain the effective potential. Corollary 1 tells us that we have found the true depolarization for  $0 \leq x \leq x_1$ ; in particular, we have found  $V(x_1, t)$ , where  $x_1$  is the first branch point.

The next step is to treat each of the sub-trees branching from  $x_1$ . Suppose, for example, that there are two daughter trees. We treat each of these separately. We collapse the upper tree into a cylinder, extending from  $x = x_1$  to  $x = L$ . The effective potential on this tree satisfies (3.4) on  $x_1 < x < L$ , with the given initial condition, the given boundary condition at the right hand end point, and the new nonhomogeneous boundary condition  $V = V(x_1, t)$  at the left hand end point. This is again a two-point boundary value problem and its solution gives the correct depolarization on the trunk

of the new tree. Now that the potential is known at the right hand end point of the trunk of the upper tree, the procedure may be repeated until the voltage over the entire upper tree is known; a similar procedure yields the depolarization on the lower tree with origin at  $x_1$ .

#### An example

We will determine the depolarization over the dendritic tree sketched in Figure 5, when an impulse current occurs at  $t=0$  at a point on the segment (2,3) which is a distance  $y$  from the origin. We assume sealed-end conditions at the origin and the dendritic terminals. For convenience let  $y > x_2$ . Note that in accordance with the notation of Theorem 1, the segments (1,2) and (2,3) both occur on the same dendrite.

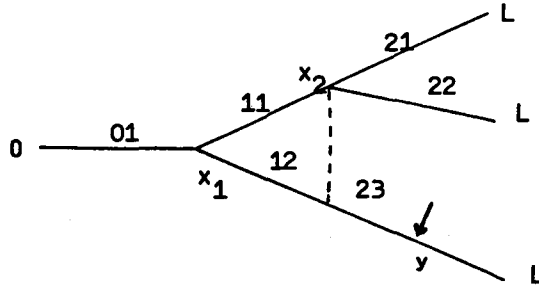


Fig. 5. Dendritic tree considered in text example. An impulse is delivered at the point  $y$  on segment (2,3).

On all segments except (2,3) the depolarization satisfies an homogeneous cable equation, with no forcing terms. On segment (2,3) we have,

$$(5.1) \quad V_{23,t} = -V_{23} + V_{23,xx} + kd_{23}^{-3/2} \delta(x-y) \delta(t),$$

where the subscripts  $t$  and  $x$  indicate partial derivatives. The effective potential, defined by (3.3), satisfies

$$(5.2) \quad V_t = -V + V_{xx} + kd_{01}^{-3/2} \delta(x-y) \delta(t), \quad 0 < x < L.$$

The solution of this equation is the Green's function given by (4.15) or (4.16):

$$(5.3) \quad V(x,t) = G(x,y;t), \quad 0 \leq x \leq L,$$

Thus the depolarization on the trunk is

$$(5.4) \quad V_{01}(x,t) = G(x,y;t), \quad 0 \leq x \leq x_1.$$

In particular, the depolarization at  $x_1$  is  $G(x_1,y;t)$ .

Consider now the upper daughter tree with its origin at  $x_1$ . This tree satisfies the equivalent cylinder assumptions, so let  $U(x,t)$  be the depolarization on this tree at

points a distance  $x$  from  $x_1$ . Then  $U$  satisfies an homogeneous cable equation on  $0 < x < L - x_1$ , with boundary conditions

$$(5.5A) \quad U(0,t) = G(x_1,y;t),$$

$$(5.5B) \quad U_x(L - x_1, t) = 0.$$

In order to obtain an equation with homogeneous boundary conditions we introduce

$$(5.6) \quad W(x,t) = U(x,t) - G(x_1,y;t),$$

so that  $W$  satisfies the equation

$$(5.7) \quad W_t = -W + W_{xx} - [G(x_1,y;t) + G_t(x_1,y;t)], \quad 0 < x < L - x_1,$$

with boundary conditions

$$(5.8A) \quad W(0,t) = 0,$$

$$(5.8B) \quad W_x(L - x_1, t) = 0.$$

Now let  $K(x,y;t)$  be the Green's function for the cable equation with a killed end condition at  $x=0$  and a sealed-end condition at  $x=L-x_1$ . (Note that  $K$  satisfies  $K_t = -K + K_{xx} + \delta(x-y) \delta(t)$ , with a coefficient of unity before the delta functions rather than  $kd^{-3/2}$ ). Then  $K$  has the Fourier series representation

$$(5.9) \quad K(x,y;t) = \frac{2H(t)e^{-t}}{L} \sum_{n=0}^{\infty} \sin[(2n+1)\pi x/2L] \sin[(2n+1)\pi y/2L] \exp[-(2n+1)^2 \pi^2 t/4L^2].$$

The solution of (5.7) with boundary conditions (5.8) is given by

$$(5.10) \quad W(x,t) = - \int_0^{L-x_1} \int_0^t K(x,z;t-s) [G(x_1,y;s) + G_s(x_1,y;s)] ds dz.$$

Evaluating the integrals we obtain an expression for  $U$ :

$$(5.11) \quad U(x,t) = \alpha H(t) e^{-t} [1 + 2 \sum_{n=1}^{\infty} \phi_n \exp(-\beta_n t)] + \frac{8\alpha H(t) e^{-t}}{\pi} \sum_{n=1}^{\infty} \frac{\psi_n(x)}{2n+1} \sum_{m=0}^{\infty} \beta_m \phi_m \left\{ \frac{e^{-\beta_m t} - e^{-\mu_n t}}{\mu_n - \beta_m} \right\}$$

where

$$(5.12) \quad \alpha = kd_{01}^{-3/2} L^{-1},$$

$$(5.13) \quad \beta_n = \frac{n^2 \pi^2}{L^2},$$

$$(5.14) \quad \mu_n = \left( \frac{(2n+1)\pi}{2(L-x_1)} \right)^2,$$

$$(5.15) \quad \phi_n = \cos\left(\frac{n\pi x_1}{L}\right) \cos\left(\frac{n\pi y}{L}\right),$$

$$(5.16) \quad \psi_n(x) = \sin\left[\frac{(2n+1)\pi x}{2(L-x_1)}\right].$$

Note that if  $\mu_n = \beta_m$  the factor in curly brackets is replaced by  $t \exp(-\mu_n t)$ . The depolarization on segments (1,1), (2,1) and (2,2) at points a distance  $x$  from  $x_1$  is given by this formula. On segment (1,2) we have, by subtraction,

$$(5.17) \quad V_{12}(x,t) = d_{12}^{-3/2} [d_{01}^{3/2} V(x,t) - d_{11}^{3/2} U(x-x_1,t)], \quad x_1 \leq x \leq x_2,$$

where  $V$  is given by (5.3) and  $U$  is given by (5.11). Similarly, on the remaining segment (2,3) we find

$$(5.18) \quad V_{23}(x,t) = d_{23}^{-3/2} [d_{01}^{3/2} V(x,t) - d_{11}^{3/2} U(x-x_1,t)], \quad x_2 \leq x \leq L.$$

The depolarization has therefore been found over the entire dendritic tree by solving cable equations with two-point boundary conditions. To illustrate, we computed the potential at the points 1-5 on the dendritic tree sketched in Figure 6A when an impulsive current occurs at the point marked with an arrow at a distance  $y=1.5$  from the origin. The values of the constants were:  $k=1$ ,  $L=2$ ,  $d_{01}=1$ ,  $d_{11}=d_{12}=d_{23}=(1/2)^{2/3}$ ,  $x_1=0.5$ ,  $x_2=1.0$ . The time courses of the potential are shown in Figure 6B.

## 6. Applications of Corollary 2

Whereas Corollary 1 is useful for neurons with single dendritic trees, Corollary 2 applies in a situation where there are several dendritic trees which have a common origin at the soma. When Corollary 2 is applied, the solution of the single cable equation (3.4) yields the depolarization at the soma for an arbitrary input.

Using arguments similar to those in section 4, it is possible to deduce, in the case of a neuron satisfying the requirements of Corollary 2, the following principle:

*The depolarization at the soma in response to a given input current occurring at any point on any dendritic tree, depends, for a given neuron, only on the distance between input location and the soma. The response is independent of which dendritic tree receives the input and is independent of the geometrical details of the tree receiving the input, and the geometrical details of all the other trees.*

It follows that the time course of the somatic depolarization depends only on the electrotonic distance from soma to input location. This observation has important implications with regard to the theoretical analysis of postsynaptic potentials, in particular those of spinal motoneurons. In such analyses, the time courses of the postsynaptic potentials generated by monosynaptic excitatory *Ia* inputs, presumably recorded at the soma, have been employed to determine the electrotonic distance from soma to synapses and to deduce the synaptic input current. Redman (1976), who references such calculations, points out that the underlying assumption was that "all

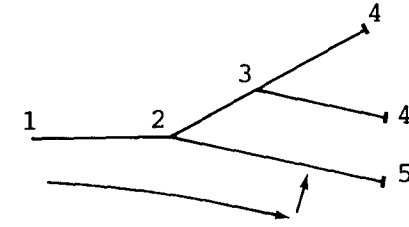


Fig. 6A. Dendritic tree for numerical example of Section 5. Impulsive current at arrow at  $t=0$ .

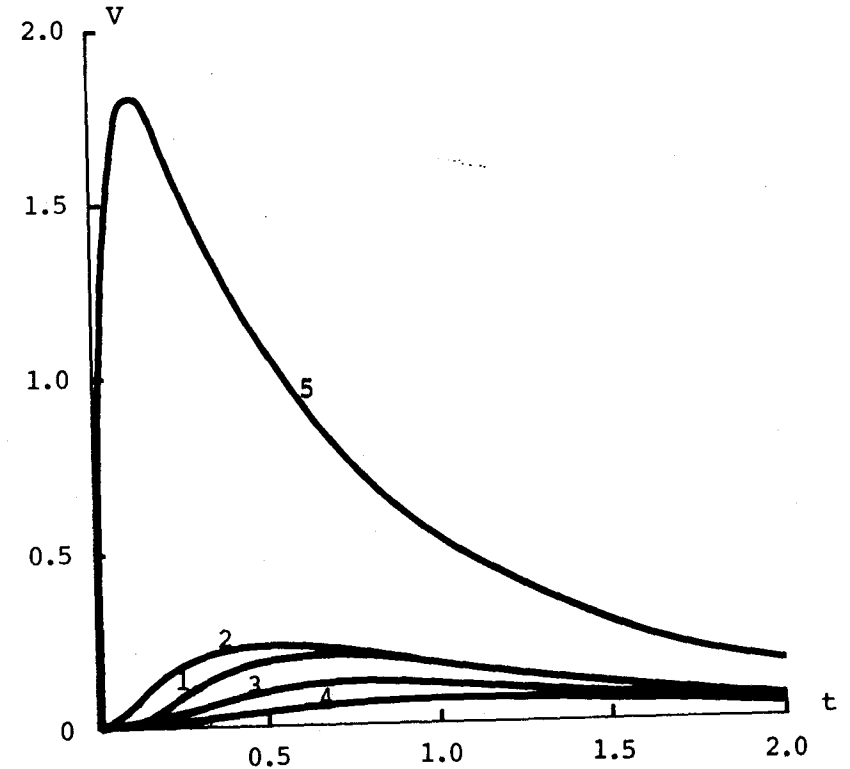


Fig. 6B. Time courses of the depolarization at points 1-5 of Figure 6A.

dendritic branches included in that cable at that electrotonic distance receive the synaptic input", but he also stated that the effects of that assumption were unknown.

As a consequence of Corollary 2, one can conclude that these previous calculations, based on the time course of the depolarization at the soma, remain valid when the requirement of synchronous activation of all dendritic branches is relaxed. The features of the postsynaptic potentials used in the previous analyses were the "half-width" and "rise-time", and these quantities do not depend on the geometrical details of the input, only its *distance* from the soma.

The application of Corollary 2 yields, on solving a single cable equation, the depolarization at the soma. The depolarization over the entire nerve cell may be found by solving a sequence of cable equations with two-point boundary conditions, as in Section 5, except that now the first step is to find the depolarization at the soma alone, rather than on the trunks of each dendritic tree.

Finally, we note that the Theorem, and Corollaries 1 and 2 may be applied in the case of random inputs (Tuckwell, 1984), though sometimes care may be needed in the handling and interpretations of the solutions.

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## Appendix

### Proof of the Theorem

We will prove this for the case where the  $I_{ij}$  are continuous functions. The more general case can be proved the same way, but requires somewhat more care.

Fix  $i$  for the moment and consider a single cylinder which branches at  $x_i$  into, say,  $m$  daughter cylinders.

Let the potential in the parent branch be  $V_{i-1,1}$  and let  $V_{i1}, \dots, V_{im}$  be the potentials in the daughter branches. Let the respective diameters be  $d_{i-1,1}, d_{i1}, \dots, d_{im}$ . Now each  $V_{pj}$  satisfies the cable equation in its interval of definition: (recall that subscripts  $t$  and  $x$  refer to partial derivatives):

$$(A.1) \quad V_{pj,t} = V_{pj,xx} - V_{pj} + kd_{pj}^{-3/2}I_{pj},$$

where (A.1) holds for  $t > 0$  and  $x_{i-1} < x < x_i$  if  $p = i-1, j = 1$ , and for  $t > 0$  and  $x_i < x < x_{i+1}$  if  $p = i$  and  $j = 1, \dots, m$ . At the branch point  $x_i$ , the continuity of potential and conservation of core current imply that

$$(A.2) \quad V_{i-1,1}(x_i^-, t) = V_{i1}(x_i^+, t) = \dots = V_{im}(x_i^+, t),$$

$$(A.3) \quad d_{i-1,1}^{3/2}V_{i-1,1}(x_i^-, t) = \sum_{j=1}^m d_{ij}^{3/2}V_{ij}(x_i^+, t).$$

Extend the functions  $V_{i-1,1}$  and  $I_{i-1,1}$  to the interval  $(x_{i-1}, x_{i+1})$  by putting

$$V_{i-1,1}(x, t) = \begin{cases} V_{i-1,1}(x_i^-, t) & \text{if } x = x_i, t > 0 \\ \sum_{j=1}^m (d_{ij}/d_{i-1,1})^{3/2}V_{ij}(x, t) & \text{if } x_i < x < x_{i+1}, t > 0. \end{cases}$$

$$I_{i-1,1}(x, t) = \begin{cases} I_{i-1,1}(x_i^-, t) & \text{if } x = x_i, t > 0 \\ \sum_{j=1}^m I_{ij}(x, t) & \text{if } x_i < x < x_{i+1}, t > 0. \end{cases}$$

Take Laplace transforms:

$$\hat{V}_{pj}(x, s) = \int_0^\infty e^{-st} V_{pj}(x, t) dt$$

$$\hat{I}_{pj}(x, s) = \int_0^\infty e^{-st} I_{pj}(x, t) dt.$$

It follows from (A.1), (A.2) and (A.3) that for each  $s > 0$

$$(A.4) \quad -\hat{V}_{i-1,1}'' + (1+s)\hat{V}_{i-1,1} = \hat{f}_{i-1,1}$$

for both  $x_{i-1} < x < x_i$  and  $x_i < x < x_{i+1}$ , where primes denote differentiation with respect to  $x$ .

$$(A.5) \quad \hat{V}_{i-1,1}(x_i^-, s) = \hat{V}_{i-1,1}(x_i^+, s) = \hat{V}_{i-1,1}(x_i, s) \doteq a(s)$$

$$(A.6) \quad \hat{V}_{i-1,1}'(x_i^-, s) = \hat{V}_{i-1,1}'(x_i^+, s) = \hat{V}_{i-1,1}'(x_i, s) \doteq b(s).$$

But now fix  $s$  and consider the function  $U(x)$  which is the unique solution of the ordinary differential equation

$$(A.7) \quad -U'' + (1+s)U = \hat{f}_{i-1,1}, \quad x_{i-1} < x < x_{i+1},$$

with initial conditions

$$(A.8) \quad U(x_i) = a(s), \quad U'(x_i) = b(s).$$

As a function of  $x$ ,  $\hat{V}_{i-1,1}(x, s)$  satisfies (A.7) and (A.8) on  $x_{i-1} < x < x_i$  and on  $x_i < x < x_{i+1}$ . Both  $U$  and  $\hat{V}_{i-1,1}$  are continuous at  $x_i$ , so they are equal on the entire interval  $x_{i-1} < x < x_{i+1}$ ; hence (A.4) holds on this interval. Inverting Laplace transforms shows that  $\hat{V}_{i-1,1}$  satisfies (A.1) on  $x_{i-1} < x < x_{i+1}$ ,  $t > 0$ .

This holds as well for the different branches  $\hat{V}_{i-1,2}, \dots, \hat{V}_{i-1, n_{i-1}}$ . Note that

$$V(x, t) = \sum_{j=1}^{n_i} (d_{i-1,j}/d_0)^{3/2} \hat{V}_{i-1,j}(x, t)$$

for all  $x_{i-1} < x < x_{i+1}$ . Thus, by linearity,  $V$  satisfies (3.4) on  $x_{i-1} < x < x_{i+1}$ , for each  $i$ . These intervals overlap so it follows that  $V$  satisfies (3.4) on the entire interval  $(0, L)$ .