On the effect of random perturbations in a nonlinear system

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There have been many articles recently on stochastic effects in nonlinear dynamical systems. One example that has been studied is the stochastic Landau equation

\[ \frac{dX}{dt} = (\lambda X - X^3)dt + \sigma dW, \quad (1) \]

where \( \lambda \) and \( \sigma \) are constants and \( \{ W(t), T \geq 0 \} \) is a standard Wiener process: that is, one with the properties that its mean is

\[ E[W(t)] = 0 \quad (2) \]

and the covariance between its values at times \( s \) and \( t \) is

\[ \text{Cov}[W(s), W(t)] = \min(s, t). \quad (3) \]

Many such studies have focused on first passage times or on steady state densities which have sometimes been estimated using electronic simulation. In some of the cases examined the noise has been multiplicative rather than additive and correlated (colored) rather than uncorrelated (white). Similar stochastic effects have been studied in several biological contexts. In Ref. 3 the nonlinear terms in (1) are derived from a potential

\[ V(x) = \frac{\lambda x^2}{2} - \frac{x^4}{4} \quad (4) \]

so \( X(t) \) can be interpreted as the speed as a function of time.

We will give a result for a general equation of the form of (1)

\[ \frac{dX}{dt} = f(X)dt + \epsilon dW, \quad (5) \]

where \( \epsilon \) is small and \( f \) is a function which has an equilibrium point \( x_0 \), which satisfies

\[ f(x_0) = 0. \]

We will demonstrate that when a dynamical system satisfying (5) is randomly perturbed about a stable (asymmetrically) equilibrium point, \( x_0 \), then the mean is shifted above or below the equilibrium point according to whether the second derivative \( f''(x_0) \), is positive or negative. Note that this occurs even though the additive noise itself has zero mean. The implication is that any attempt to measure \( x_0 \) by averaging will lead to an estimate which is shifted away from the true value, and in particular, is shifted up or down depending on whether \( f''(x_0) \) is positive or negative.

To see this we expand \( X(t) \) in powers of \( \epsilon \) about the initial value which is assumed to be at an equilibrium point, i.e., \( X(0) = x_0 \);

\[ X = x_0 + \sum_{k=1}^{\infty} \epsilon^k X_k. \quad (6) \]

Substituting in (5) and equating coefficients of powers of \( \epsilon \) we obtain a sequence of linear equations whose first two members are

\[ \begin{align*}
X_1 &= f'(x_0)X_1 + \sigma W, \\
X_2 &= f'(x_0)X_2 + \frac{f''(x_0)X_1^2}{2!}. \tag{7}
\end{align*} \]

Now because \( x_0 \) is assumed to be stable, \( f''(x_0) \) is negative. Hence, the solution of (7) is a classical Ornstein-Uhlenbeck process and we may write

\[ X_1(t) = X_1(0)e^{\sigma t} + \int_0^t e^{\sigma(t-s)}dW(s), \quad (9) \]

where we have put \( f''(x_0) = f''_0 \). Now, because the mean of integral in (9) is zero,

\[ E[X_1(t)] = E[X_1(0)]e^{\sigma t}. \quad (10) \]

However, \( X(0) = x_0 \) making \( E[X_1(0)] = 0 \) so it follows that \( E[X_1(t)] = 0 \) for \( t > 0 \).

Similarly, the solution of (8) is

\[ X_2(t) = X_2(0)e^{\sigma t} + \frac{f''_0}{2!} \int_0^t e^{\sigma(t-s)}X_1^2(s)ds, \quad (11) \]

where \( f''_0 = f''(x_0) \). Now \( E[X_2(0)] = 0 \) and \( X_1^2(t) \) is always positive for \( t > 0 \). Thus, \( E[X_2(t)] \) has the same sign as \( f''_0 \) which is sufficient to establish the stated result. However, we can find an exact result simply by evaluating \( E[X_2(t)] \) and integrating in (11). This gives

\[ E[X_2(t)] = \frac{f''_0 e^{-\alpha t}}{2\alpha^2} [\cosh(\alpha t) - 1], \quad (12) \]

where \( \alpha = |f''_0| \). Thus, we obtain from the expansion (6),

\[ E[X(t)] = x_0 + \frac{\epsilon^2 f''_0 e^{-\alpha t}}{2\alpha^2} [\cosh(\alpha t) - 1] + O(\epsilon^3). \quad (13) \]

Since all the factors that multiply \( f''_0 \) are positive, we see that to order \( \epsilon^2 \),

\[ E[X(t)] > x_0 \quad \text{if} \quad f''_0 > 0, \]

whereas

\[ E[X(t)] < x_0 \quad \text{if} \quad f''_0 < 0, \]

which is the required result.

The result remains true in the steady state because

\[ \cosh(\alpha t) \rightarrow \frac{e^{\alpha t}}{2} \quad \text{so} \]

\[ E[X(t)] = x_0 + \frac{\epsilon^2 f''_0}{2\alpha^2} [\frac{e^{\alpha t}}{2} - 1] + O(\epsilon^3). \]
$E[X(t)] \rightarrow x_0 + \frac{e^{2f''_0}}{4\alpha^2} + O(e^3)$.

One concludes, therefore, that a measurement of $x_0$ in the presence of noise, here assumed white and additive, obtained by averaging over several readings may lead to an error in the estimate of $x_0$, despite the fact that the additive noise has itself a mean value of zero.

In the case of the stochastic Landau equation (1), the stable equilibria occur at $x_0 = \mp \sqrt{\lambda}$. For random perturbations about these two equilibrium points we have, since $f'_0(x_0) = -2\lambda$ and $f''_0(x_0) = 6\sqrt{\lambda}$, on substituting in (13),

$$E[X(t)] = x_0 + \frac{3e^2e^{-2\lambda t}}{4\lambda^{3/2}} [\cosh(2\lambda t) - 1] + O(e^3).$$

Hence, at the upper equilibrium point the mean is displaced downwards, whereas at the lower equilibrium point the mean is displaced upwards, which decreases the mean distance between the equilibria.

5. F. Moss and P. V. E. McClintock, in Ref. 4.