On the effect of random perturbations in a nonlinear system

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(Received 28 April 1992; accepted 14 July 1992)

There have been many articles recently on stochastic effects in nonlinear dynamical systems.[1] One example that has been studied is the stochastic Landau equation

$$dX=(\lambda X-X^3)dt+\sigma dW, \tag{1}$$

where λ and σ are constants and $\{W(t),T\geqslant 0\}$ is a standard Wiener process: that is, one with the properties that its mean is

$$E[W(t)]=0 \tag{2}$$

and the covariance between its values at times s and t is

$$\mathrm{Cov}[W(s),W(t)]=\min(s,t). \tag{3}$$

Many such studies have focused on first passage times[2,3] or on steady state densities which have sometimes been estimated using electronic simulation.[4] In some of the cases examined the noise has been multiplicative rather than additive and correlated (colored) rather than uncorrelated (white). Similar stochastic effects have been studied in several biological contexts.[5–8] In Ref. 3 the nonlinear terms in (1) are derived from a potential

$$V(x)=\frac{\lambda x^2}{2}-\frac{x^4}{4} \tag{4}$$

so $X(t)$ can be interpreted as the speed as a function of time.

We will give a result for a general equation of the form of (1)

$$dX=f(X)dt+\epsilon dW, \tag{5}$$

where ϵ is small and f is a function which has an equilibrium point x_0, which satisfies

$$f(x_0)=0.$$

We will demonstrate that when a dynamical system satisfying (5) is randomly perturbed about a stable (asymptotically) equilibrium point, x_0, then the mean is shifted above or below the equilibrium point according to whether the second derivative $f''(x_0)$, is positive or negative. Note that this occurs even though the additive noise itself has zero mean. The implication is that any attempt to measure x_0 by averaging will lead to an estimate which is shifted away from the true value, and in particular, is shifted up or down depending on whether $f''(x_0)$ is positive or negative.

To see this we expand $X(t)$ in powers of ϵ about the initial value which is assumed to be at an equilibrium point, i.e., $X(0)=x_0$;

$$X=x_0+\sum_{k=1}^{\infty}\epsilon^k X_k. \tag{6}$$

Substituting in (5) and equating coefficients of powers of ϵ we obtain a sequence of linear equations whose first two members are

$$dX_1=f'(x_0)X_1dt+dW, \tag{7}$$

$$dX_2=\left[f'(x_0)X_2+\frac{f''(x_0)X_1^2}{2!}\right]dt. \tag{8}$$

Now because x_0 is assumed to be stable, $f'(x_0)$ is negative. Hence, the solution of (7) is a classical Ornstein–Uhlenbeck process[9] and we may write

$$X_1(t)=X_1(0)e^{f_0't}+\int_0^t e^{f_0'(t-s)}dW(s), \tag{9}$$

where we have put $f'(x_0)=f_0'$. Now, because the mean of integral in (9) is zero,

$$E[X_1(t)]=E[X_1(0)]e^{f_0't}. \tag{10}$$

However, $X(0)=x_0$ making $E[X_1(0)]=0$ so it follows that $E[X_1(t)]=0$ for $t\geqslant 0$.

Similarly, the solution of (8) is

$$X_2(t)=X_2(0)e^{f_0't}+\frac{f_0''e^{f_0't}}{2!}\int_0^t e^{-f_0's}X_1^2(s)ds, \tag{11}$$

where $f_0''=f''(x_0)$. Now $E[X_2(0)]=0$ and $X_1^2(t)$ is always positive for $t>0$. Thus, $E[X_2(t)]$ has the same sign as f_0'' which is sufficient to establish the stated result. However, we can find an exact result simply by evaluating $E[X_1^2(t)]$ and integrating in (11). This gives

$$E[X_2(t)]=\frac{f_0''e^{-\alpha t}}{2\alpha^2}[\cosh(\alpha t)-1], \tag{12}$$

where $\alpha=|f_0'|$. Thus, we obtain from the expansion (6),

$$E[X(t)]=x_0+\frac{\epsilon^2 f_0''e^{-\alpha t}}{2\alpha^2}[\cosh(\alpha t)-1]+O(\epsilon^3). \tag{13}$$

Since all the factors that multiply f_0'' are positive, we see that to order ϵ^2,

$$E[X(t)]>x_0 \quad \text{if} \quad f_0''>0,$$

whereas

$$E[X(t)]<x_0 \quad \text{if} \quad f_0''<0,$$

which is the required result.

The result remains true in the steady state because $\cosh(\alpha t)\rightarrow\frac{e^{\alpha t}}{2}$, so

$$E[X(t)]\to x_0+\frac{\epsilon^2 f_0''}{4\alpha^2}+O(\epsilon^3).$$

One concludes, therefore, that a measurement of x_0 in the presence of noise, here assumed white and additive, obtained by averaging over several readings may lead to an error in the estimate of x_0, despite the fact that the additive noise has itself a mean value of zero.

In the case of the stochastic Landau equation (1), the stable equilibria occur at $x_0=\pm\sqrt{\lambda}$. For random perturbations about these two equilibrium points we have, since $f_0'(x_0)=-2\lambda$ and $f_0''(x_0)=\mp 6\sqrt{\lambda}$, on substituting in (13),

$$E[X(t)]=x_0\mp\frac{3\epsilon^2 e^{-2\lambda t}}{4\lambda^{3/2}}[\cosh(2\lambda t)-1]+O(\epsilon^3).$$

Hence, at the upper equilibrium point the mean is displaced downwards, whereas at the lower equilibrium point the mean is displaced upwards, which decreases the mean distance between the equilibria.

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