ON THE FIRST-EXIT TIME PROBLEM FOR TEMPORALLY HOMOGENEOUS MARKOV PROCESSES

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Abstract

Using an integral equation of Darling and Siegert in conjunction with the backward Kolmogorov equation for the transition probability density function, recurrence relations are derived for the moments of the time of first exit of a temporally homogeneous Markov process from a set in the phase space. The results, which are similar to those for diffusion processes, are used to find the expectation of the time between impulses of a Stein model neuron.

FIRST-EXIT TIMES; MARKOV PROCESSES; RECURRENCE RELATIONS; NEURON

1. Introduction

The first-passage time density for the Wiener process has been known for a long time (see Siegert (1951) for historical references). For such a process, with initial value \( x_0 \), the density of the time, \( T_t(x^*) \), to first reach \( x^* \) is, with \( g(t, x^* | x_0) \) \( dt = P (T_t(x^*) \in (t, t + dt) | X(0) = x_0) \),

\[
(1) \quad g(t, x^* | x_0) = \frac{|x^* - x_0|}{\sqrt{2\pi Vt^3}} \exp \left[ -\frac{(x^* - x_0 - mt)^2}{2Vt} \right],
\]

where it is assumed that the process has drift \( mt \) and variance parameter \( V \), and furthermore that the drift is in the direction from \( x_0 \) to \( x^* \), otherwise passage through \( x^* \) is not certain and the first-passage time has no finite moments.

The Wiener process is a highly specialized random function, and when one considers the first-passage time problem for continuous processes in the general case, one finds a paucity of analytic results. Siegert (1951) found the following formula for the first-passage time density when \( x^* \) is a symmetry point of the process:

\[
(2) \quad g(t, x^* | x_0) = \frac{a}{\partial t} \int_{-\infty}^{x^*} p(x, t | x_0) \, dx,
\]

where \( p(x, t | x_0) \) is the transition density of the unrestricted process, assumed temporally homogeneous. A special case of this formula, for the velocity of a

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Brownian motion particle in the Ornstein–Uhlenbeck model, was obtained by Wang and Uhlenbeck (1945).

Also worked out by Siegert (1951) was a method of studying first-passage time distributions for continuous processes by means of the Laplace transforms of their densities. The starting point was the renewal equation,

\[
p(x, t \mid x_0) = \int_0^t g(t', x^* \mid x_0) p(x, t - t' \mid x^*) \, dt'
\]

which requires that \(x_0 < x^* \leq x\). This approach was further developed by Darling and Siegert (1953) who also obtained the following result.

**Theorem 1** (Darling and Siegert). Let \(X(t)\) be a diffusion process with time-independent first and second infinitesimal moments \(K_1\) and \(K_2\) respectively, with \(X(0) = x\) and \(x \in (x_1, x_2)\). If

\[
T = \sup \{t \mid X(t') \in (x_1, x_2) \forall t' \in [0, t]\},
\]

is a proper random variable, then its moments, denoted by \(t_n, n = 0, 1, 2, \ldots\), satisfy the recurrence relations

\[
\frac{1}{2} K_2(x) \frac{d^2 t_n(x)}{dx^2} - K_1(x) \frac{dt_n(x)}{dx} = -nt_{n-1}(x),
\]

with \(t_n(x_1) = t_n(x_2) = 0, n = 1, 2, \ldots\).

We mention this theorem because we will obtain its generalization to the case where the sample paths of the process may have discontinuities of the first kind. For such processes it is clear that the renewal equation (3) cannot always be written down, because sample paths do not have to pass through all values intermediate between the initial and final values.

The first-passage time problem for such processes has received little attention except for those with independent increments which are also temporally homogeneous. One investigation, that of Keilson (1963), used an elegant method (the method of compensation) to find the following expression for the density of the first passage time when the considered process is also spatially homogeneous:

\[
g(t, 0 \mid x_0) = \frac{x_0}{t} p(-x_0, t \mid 0).
\]

However this result is only applicable when the value zero is approached via smooth (skip-free) trajectories.

Finally we mention that other studies of first-passage problems have been
performed by Zolotarev (1964), Borovkov (1965), Rogozin (1965), Shtatland (1965) and Gusak and Koralyuk (1968), all of whom utilized the general expression for the characteristic function of a temporally homogeneous independent increment process.

We will derive, using analytic techniques, the recurrence relations satisfied by the moments of the time of first exit from a given set for a temporally homogeneous process satisfying a stochastic differential equation. The equations for the first two moments have been derived using different techniques by Gihman and Skorohod (1972). Our work is an extension of that of Chuang (1970).

2. Derivation of the recurrence relations for the moments

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\Theta$ be the continuous parameter set $[0, \infty)$. Suppose that a random process $X(\omega, t)$ is defined on $\Omega \times \Theta$, where $\omega \in \Omega$, $t \in \Theta$, and assume that it takes values in $\mathbb{R}$ (the real line). We will further suppose that $X(\omega, t)$ is the solution of the stochastic differential equation

\[ dX(t) = \alpha(X(t))dt + \beta(X(t))dW(t) \]
\[ - \int_\mathbb{R} \gamma(X(t), u)\nu(dt \times du), \]

where reference to the underlying probability space is omitted, $W(t)$ is the standard Wiener process and $\nu(t, \cdot)$ is a temporally homogeneous Poisson random measure. Providing existence and uniqueness conditions are fulfilled, the solution of (6) will be a temporally homogeneous Markov process,

\[ X(t) = X(0) + \int_0^t \alpha(X(t'))dt' + \int_0^t \beta(X(t'))dW(t') \]
\[ + \int_0^t \int_\mathbb{R} \gamma(X(t'), u)\nu(dt' \times du), \]

where $X(0)$ is the initial value of $X(t)$. For the definitions of the integrals with respect to the Wiener process and the Poisson measure together with certain existence and uniqueness theorems, see Gihman and Skorohod (1972).

We now further assume that conditions are satisfied for the transition probability of $X(t)$ to have a density, $p(y, t_2 | x, t_1)$, with $t_1 < t_2$. Using the Ito definition of stochastic integral with respect to $W(t)$, the last-mentioned authors have shown that this density satisfies the backward Kolmogorov equation
\[- \frac{\partial p}{\partial t_1} = - \Lambda p + \alpha(x) \frac{\partial p}{\partial x} + \frac{1}{2} \beta(x)^2 \frac{\partial^2 p}{\partial x^2} \]

\[+ \int_R p(y, t_1 | x + \gamma(x, u), t_1) \Pi(du), \]

where \( \Pi(\cdot) \) is the measure defined on \( \mathcal{B}(R) \) such that

\[E[\nu(t, B)] = t \Pi(B), \quad B \in \mathcal{B}(R), \]

and the jump intensity,

\[\Lambda = \int_R \Pi(du), \]

is assumed to be finite. We now prove the following result.

**Theorem 2.** Let \( X(t) \) be the temporally homogeneous process whose transition probability density satisfies (8). Let \( A \) be an open set in the phase space and define, for \( X(0) = x \in A \),

\[T_A(x) = \inf\{t | X(t) \notin A\} \]

which is the time of first exit from \( A \). We set

\[M_A^n(x) = E[T_A(x)^n], \quad n = 0, 1, 2, \ldots, \]

if these quantities exist. We then have the following.

(i) The probability \( M_A^0(x) \) that \( X(t) \) ever leaves the set \( A \) satisfies the equation

\[- \Lambda M_A^0(x) + \alpha(x) \frac{dM_A^0(x)}{dx} + \frac{1}{2} \beta(x)^2 \frac{d^2M_A^0(x)}{dx^2} \]

\[+ \int_R M_A^0(x + \gamma(x, u)) \Pi(du) = 0, \]

with boundary conditions \( M_A^0(x) = 1, x \notin A \).

(ii) If the solution of (12) is \( M_A^0(x) = 1 \) for all \( x \in A \), then the moments of \( T_A(x) \) satisfy the recurrence relations

\[- \Lambda M_A^n(x) + \alpha(x) \frac{dM_A^n(x)}{dx} + \frac{1}{2} \beta(x)^2 \frac{d^2M_A^n(x)}{dx^2} \]

\[+ \int_R M_A^0(x + \gamma(x, u)) \Pi(du) = - n M_A^{n-1}(x), \quad n = 1, 2, \ldots, \]

with boundary conditions \( M_A^0(x) = 0 \) for \( x \notin A \).
Proof. Given that \( p(y, t_2 \mid x, t_1) \) satisfies (8) it will be convenient to introduce the operator \( L_0^+ \) through

\[
-\frac{\partial p}{\partial t_1} = L_0^+ p.
\]

Consider now the random functional

\[
U(t_1, t_2) = \int_{t_1}^{t_2} \Phi[X(t'), t'] dt',
\]

where \( \Phi(\cdot, \cdot) \) is defined on \( \mathbb{R} \times \Theta \) and takes non-negative values. Define

\[
r(x, t_1 \mid y, t_2, \mu) = E[e^{-\mu(t_2-t_1)} \mid X(t_1) = x, X(t_2) = y \mid p(y, t_2 \mid x, t_1)].
\]

Then, \( X(t) \) being Markov, Darling and Siegert (1957) have shown that \( r \) satisfies the integral equation

\[
r(x, t_1 \mid y, t_2, \mu) = p(y, t_2 \mid x, t_1) - \mu \int_{t_1}^{t_2} dt' dx' p(x', t' \mid x, t_1) \Phi(x', t') r(x, t_1 \mid x', t'; \mu).
\]

If we operate throughout this equation with \( L_0^+ + \partial / \partial t_1 \), we find, using (8),

\[
-\frac{\partial r}{\partial t_1} = [L_0^+ - \mu \Phi(x, t_1)] r(x, t_1 \mid y, t_2, \mu).
\]

The Laplace transform of \( U(t_1, t_2) \),

\[
\phi^*(x, t_1, t_2, \mu) = \int_{-\infty}^{\infty} r(x, t_1 \mid y, t_2, \mu) dy
\]

will thus satisfy

\[
(L_0^+ + \frac{\partial}{\partial t_1}) \phi^*(x, t_1, t_2; \mu) = \mu \Phi(x, t_1) \phi^*(x, t_1, t_2; \mu).
\]

Let \( A \) be an open set in the phase space of \( X(t) \) and define the function \( \Phi(X(t), t) \) by

\[
\Phi(X(t), t) = \begin{cases} 0, & \text{if } X(t) \in A, \\ 1, & \text{if } X(t) \notin A, \end{cases}
\]

so that the corresponding transform \( \phi^*_A \) satisfies

\[
(L_0^+ + \frac{\partial}{\partial t_1}) \phi^*_A(x, t_1, t_2; \mu) = 0, \quad x \in A.
\]
The probability $P_A(x, t_1, t_2)$ that $X(t)$ remains in $A$ throughout the entire time interval $[t_1, t_2]$, given that $X(0) = x \in A$, is given by

$$P_A(x, t_1, t_2) = \lim_{\mu \to \infty} \phi^\star_A(x, t_1, t_2; \mu).$$

(23)

Hence this quantity satisfies

$$\left( L^\star + \frac{\partial}{\partial t_1} \right) P_A(x, t_1, t_2) = 0, \ x \in A,$$

(24)

with the initial condition

$$P_A(x, t_1, t_1) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

(25)

and boundary conditions

$$P_A(x, t_1, t_2) = 0, \ x \notin A.$$

(26)

To simplify the notation we set $t_2 - t_1 = \tau$, and utilize temporal homogeneity by putting $P_A = P_A(x, \tau)$, which will satisfy

$$\frac{\partial P_A}{\partial \tau} = L^\star P_A(x, \tau).$$

(27)

Now $P_A(x, \tau) = P(T_A(x) > \tau \geq 0) = 1 - P(0 \leq T_A(x) \leq \tau) = 1 - F_A(x, \tau)$ where $F_A(x, \tau)$ is the distribution function of $T_A(x)$. Clearly this is determined by

$$\frac{\partial F_A}{\partial \tau} = L^\star F_A(x, \tau), \ x \in A,$$

(28)

with initial condition

$$F_A(x, 0) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \notin A, \end{cases}$$

(29)

and boundary conditions $F_A(x, \tau) = 1$ if $x \notin A$. The Laplace–Stieltjes transform of $F_A$, defined through

$$F^\star_A(x, s) = \int_{[0, \infty)} e^{-s\tau} F_A(x, d\tau)$$

(30)

is found, from (29), to satisfy

$$sF^\star_A(x, s) = L^\star F^\star_A(x, s), \ x \in A,$$

(31)

with boundary conditions $F^\star_A(x, s) = 0$ for $x \notin A$. Utilizing the relation

$$F^\star_A(x, s) = \sum_{n=0}^\infty \frac{(-s)^n}{n!} M^{(n)}_A(x),$$

(32)
and substituting in (31) we obtain the recurrence relations
\begin{equation}
L^\circ_m M^{(n)}_A(x) = -nM^{(n-1)}_A(x), \quad x \in A, \quad n = 1, 2, \ldots.
\end{equation}
In particular, \(M^{(0)}_A(x)\), the probability that \(X(t)\) ever leaves \(A\), satisfies
\begin{equation}
L^\circ_0 M^{(0)}_A(x) = 0, \quad x \in A
\end{equation}
with the boundary conditions \(M^{(0)}_A(x) = 1\) for \(x \notin A\).

If the solution of (34) is \(M^{(n)}_A(x) = 1\) for all \(x \in A\), then the moments of \(T_A(x)\) are obtained from (33) for \(n = 1, 2, \ldots\), with boundary conditions \(M^{(n)}_A(x) = 0\) for \(x \notin A\). If we write out (33) using the definition of \(L^\circ_m\) given by (8) and (14), we obtain the desired recurrence relations for the moments of \(T_A(x)\). The equations with \(n = 1\) and \(n = 2\) are identical to those obtained by Gihman and Skorohod (1972) from the generalized Itô formula.

3. Application to Stein's model neuron

As an illustration of the usefulness of Theorem 2 we consider a model for the activity of nerve membrane depolarization proposed by Stein (1965). The basic physiological ideas are that the depolarization potential, \(X(t)\), increases by steps of magnitude \(\varepsilon\) according to a Poisson process of parameter \(\lambda\), and that between steps (excitatory post-synaptic potentials or epsp's) the depolarization decays to the resting level of zero according to
\begin{equation}
\frac{dX(t)}{dt} = -\sigma X(t),
\end{equation}
where \(\sigma\) is the reciprocal of the time constant of the membrane. As initial condition one has \(X(0) = 0\) and the problem is to determine the random variable \(\tau\) which denotes the time at which \(X(t)\) first reaches or exceeds some (fixed) value \(\theta\) which is called the threshold for firing. After the firing has occurred, \(X(t)\) is reset to zero for the duration of the absolutely refractory period whereupon the excitation is 'switched on' again and the random process \(X(t)\) starts anew.

The stochastic process \(X(t)\), representing the depolarization due to the combined effects of randomly occurring epsp's and the exponential decay, satisfies the stochastic differential equation
\begin{equation}
dX(t) = -\sigma X(t)dt + \int f(u) \nu (dt \times du)
\end{equation}
where \(f(u) = \varepsilon\) for \(u\) in some set \(\Delta_\varepsilon\) whose \(\Pi\)-measure is \(\lambda\) and \(f(u) = 0\) otherwise. It is clear that \(X(t)\) is Markov and not independent-increment. Thus whereas Theorem 2 may be applied to independent-increment processes (e.g.
the Poisson process or the Poisson process superimposed upon a positive linear drift) we wish to show here that new results may be obtained for other than independent-increment processes. We point out that a more complete discussion of the application to the problem of determining the inter-spike time of a neuron undergoing random synaptic excitation is to appear elsewhere (Tuckwell (1975)), and that an account of the physiological concepts can be obtained from Eccles (1964).

Let \( A \) be the set \((\Delta, \theta)\) and suppose \( \Delta < \varepsilon < \theta \). The occurrence of the first epsp will take the depolarization to \( \varepsilon \), and we will designate the time of occurrence as \( \tau_1 \) (which is a known random variable). We now consider the ‘new’ process starting at \( \varepsilon \in A \). We let \( t_\lambda(x) \) denote the time of first exit from \( A \) for an initial value \( x \). Hence we may write

\[
\tau = \tau_1 + \lim_{\Delta \downarrow 0} t_\lambda(\varepsilon),
\]

which will enable us to find the moments of the time between nerve firings.

From Equations (12), (13) and (36) we find that the moments

\[
E[t_\lambda(x)^n] = M^{(n)}_{\lambda}(x),
\]

satisfy the differential-difference equations

\[
-\sigma x \frac{dM^{(n)}_{\lambda}(x)}{dx} + \lambda [M^{(n)}_{\lambda}(x + \varepsilon) - M^{(n)}_{\lambda}(x)] = -nM^{(n-1)}_{\lambda}(x),
\]

where \( n = 1, 2, \ldots \), and the boundary conditions are as given in the previous section.

It is clear by substitution that unity is a solution of the equation for the probability that the process ever leaves \( A \),

\[
-\sigma x \frac{dM^{(0)}_{\lambda}(x)}{dx} + \lambda [M^{(0)}_{\lambda}(x + \varepsilon) - M^{(0)}_{\lambda}(x)] = 0,
\]

where \( M^{(0)}_{\lambda}(x) = 1 \) for \( x \notin A \). A more detailed proof that this is the solution, based on the assumptions of continuity on \([\Delta, \theta]\) and differentiability on \( A \), has been given in our previously mentioned article. Thus the moments of \( t_\lambda(x) \) are finite and the recurrence relations (39) can be employed, for \( n = 1, 2, 3, \ldots \), to find these moments. In the limit \( \Delta \downarrow 0 \) the solution of (40) is still \( M^{(0)}_{\lambda}(x) = 1 \): that is, the moments are still finite and in this limiting case, since the depolarization cannot escape from \( A \) at the lower boundary in a finite time (exponential decay), the threshold for firing must be attained in a finite time with probability one.

We will find the expectation of the time that the depolarization first reaches or exceeds threshold in a simple case which is of physiological interest. Firstly we point out that the moments of the firing time, in Stein’s model for random
excitation, depend only on the ratio of the threshold $\theta$ to the epsp magnitude $\epsilon$. This so-called scale invariance is due to the structure of the differential-difference equations (39). Without loss of generality, therefore, we may set $\epsilon = 1$ and re-interpret $\theta$ as the ratio of threshold depolarization to epsp magnitude. We consider the case $\theta = 1 + \Delta_i$ where $\Delta_i \leq 1$ and take the set $A$ to be $(\Delta, 1 + \Delta_i)$ with $\Delta < 1$.

Let us now denote the expectation of the time of first exit from $A$ by $F(x)$ where $x \in A$. The equation for $F(x)$,

\begin{equation}
-\sigma x \frac{dF}{dx} + \lambda [F(x + 1) - F(x)] = -1,
\end{equation}

may be solved by proceeding from the upper boundary of $A$ in steps of unit length according to the formula

\begin{equation}
F(x) = x^{-\lambda/\sigma} \left[ \frac{x^{\lambda/\sigma}}{\lambda} - \frac{\lambda}{\sigma} \int_0^x y^{(\lambda/\sigma)-1}F(y + 1)dy + \text{const.} \right].
\end{equation}

We will also set $\lambda = n\sigma$, $n$ being a positive integer, in order to facilitate finding analytic results. On $[\Delta_i, 1 + \Delta_i]$ we write $F(x) = F_i(x)$ which is simply

\begin{equation}
F_i(x) = x^{-a_1} \left[ \frac{x^n}{\lambda} + a_1 \right],
\end{equation}

where $a_1$ is the first constant of integration. On $(\Delta, \Delta_i)$ we set $F(x) = F_2(x)$ which, after substituting (43) in (42), is found to be

\begin{equation}
F_2(x) = x^{-n} \left[ \frac{2x^n}{\lambda} + na_1 \left[ \log(x + 1) - \sum_{s=1}^{n-1} k_{ns}(x + 1)^{-s} \right] + a_2 \right],
\end{equation}

where $a_2$ is the second integration constant, and we have defined

\begin{equation}
k_{ns} = \binom{n-1}{s} \left( \frac{-1}{s} \right)^r.
\end{equation}

We note that the summation in (44) will be absent unless $n > 1$.

To find the constants of integration we impose the conditions that $F(x)$ be continuous at $\Delta_i$ and that $F(x)$ continuously approach zero from above as $x \downarrow \Delta$. This gives two equations in $a_1$ and $a_2$. We can obtain the expectation of the time to first exit starting from $x = 1$ from a knowledge of $a_1$ which is given by

\begin{equation}
a_1 = \frac{1}{\lambda} \frac{(\Delta_i^r - 2\Delta^n)}{\left( 1 + n \left( \log \frac{1 + \Delta}{(1 + \Delta_i)} + \sum_{s=1}^{n-1} k_{ns} [(1 + \Delta_i)^{-s} - (1 + \Delta)^{-s}] \right) \right)}.
\end{equation}

Since $E(\tau_1) = 1/\lambda$, we may then find $E(\tau)$ from Equation (37) with $\epsilon = 1$. In the simplest case, $n = 1$, this gives
\[
E(\tau) = \frac{1}{\lambda} \left[ 2 + \frac{\Delta_i}{1 - \log(1 + \Delta_i)} \right].
\]

In our previous report (Tuckwell (1975)) the case \(\Delta_i = 1\) was used to determine the output frequency of cat motoneurons undergoing Poisson synaptic excitation. At a mean input frequency of 102 sec\(^{-1}\) the calculated frequency of firing is 9.7 sec\(^{-1}\) (employing an absolute refractory period of 1.5 msec) whereas the experimental value of Redman et al. (1968) is 9.8 sec\(^{-1}\). Below the cited input frequency the assumption \(\Delta_i = 1\) leads to results which are less than the experimentally obtained output frequencies.

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References


