Use of Green’s function matrices for systems of diffusion equations

HENRY C. TUCKWELL†

The usefulness of Green’s function matrices for multi-dimensional diffusion equations is indicated. A general result is obtained and an application to two coupled cable equations is made.

1. Introduction


A useful starting point for analysing non-linear systems of reaction-diffusion equations is found through linear systems obtained by expanding the non-linear terms about some critical point. It is the purpose of this paper to show that Green’s function matrices are as useful for multi-component systems as scalar Green’s functions are for one-dimensional problems.

2. Green’s function matrices

We focus attention on multi-dimensional diffusion equations but bear in mind that applications can also be made to elliptic and hyperbolic equations. In particular, let \( \mathbf{u} = \mathbf{u}(x, t) \) be an \( n \times 1 \) vector, \( \mathbf{D} \) an \( n \times n \) constant diagonal matrix with positive elements \( D_i \), for \( i = 1, \ldots, n \), \( \mathbf{A} \) an \( n \times n \) constant matrix, and \( \mathbf{f} = \mathbf{f}(x, t) \) an \( n \times 1 \) vector. Letting subscripts \( x \) and \( t \) denote partial derivatives with respect to these variables, we are interested in finding solutions of the system of \( n \) equations

\[
\mathbf{u}_t = \mathbf{D} \mathbf{u}_{xx} + \mathbf{A} \mathbf{u} + \mathbf{f}, \quad a < x < b, \quad t > 0
\]  

with initial data

\[
\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad a \leq x \leq b
\]

and homogeneous linear boundary conditions of the form

\[
\begin{align*}
\alpha_1 \mathbf{u}(a, t) + \beta_1 \mathbf{u}_x(a, t) &= \mathbf{0}, \quad t \geq 0 \\
\alpha_2 \mathbf{u}(b, t) + \beta_2 \mathbf{u}_x(b, t) &= \mathbf{0}, \quad t \geq 0
\end{align*}
\]

Let \( G_1(x, y; t) \) be the Green’s function for the scalar heat equation, so that \( G_1 \) satisfies

\[
G_1 = G_{xx}, \quad a < x < b, \quad t > 0
\]

with \( G(x, y; 0) = \delta(x - y) \) and suitable boundary conditions at \( x = a \) and \( x = b \). Then it is well-known that the solution of

\[
u_t = u_{xx} + h(x, t), \quad a < x < b, \quad t > 0
\]
is
\[ u(x, t) = \int_a^b G_1(x, y; t)u(y, 0) \, dy + \int_a^b \int_0^t G_1(x, y; t-s)h(y, s) \, ds \, dy \] (6)

Here it is assumed that \( h \) is such that a unique solution of (5) exists, and in the following we assume that \( f \) is such that (1)−(3) have a unique solution. The extension of (6) to represent the solution of (1)−(3) is then as follows.

**Theorem**

Let \( G(x, y; t) \) be the solution of
\[ G_t = D G_{xx} + AG, \quad a < x < b, \quad t > 0 \]
with \( G(x, y; 0) = 1\delta(x - y) \), where \( I \) is the \( n \times n \) identity matrix, and
\[ \begin{align*}
\alpha_1 G(a, y; t) + \beta_1 G_x(a, y; t) &= 0 \\
\alpha_2 G(b, y; t) + \beta_2 G_x(b, y; t) &= 0
\end{align*} \] (7)

Then the solution of (1)−(3) is given by
\[ u(x, t) = \int_a^b G(x, y; t)u_0(y) \, dy + \int_a^b \int_0^t G(x, y; t-s)f(y, s) \, ds \, dy \] (8)

**Proof**

The proof proceeds as in the scalar case, that is, one checks that \( u(x, t) \) given by (8) satisfies:

(a) the differential equations (1);
(b) the initial condition (2); and
(c) the boundary conditions (3).

These are routine calculations and need not be given in detail. \( \square \)

When the diffusion coefficients in each of the component equations are equal it is a simple matter to find the matrix \( G(x, y; t) \) which we call the Green's function matrix.

**Corollary**

If
\[ D = I \] (9)

then the Green's function matrix is given by
\[ G(x, y; t) = \exp(At) \, G_1(x, y; t) \] (10)

where \( G_1 \) is the Green's function for the scalar heat equation as in (4), that satisfies the boundary conditions
\[ \begin{align*}
\alpha_1 G_1(a, y; t) + \beta_1 G_{1,x}(a, y; t) &= 0 \\
\alpha_2 G_1(b, y; t) + \beta_2 G_{1,x}(b, y; t) &= 0
\end{align*} \] (11)
Proof

When \( \mathbf{D} = \mathbf{I} \) the substitution

\[
\mathbf{u}(x, t) = \exp(-\mathbf{A}t)\mathbf{u}(x, t)
\]

gives the uncoupled system

\[
\mathbf{v}_t = \mathbf{v}_{xx} + \mathbf{g}(x, t)
\]

where \( \mathbf{g} = \exp(-\mathbf{A}t)\mathbf{f} \). It is clear from the scalar case that the solution of (13) is

\[
\mathbf{v}(x, t) = \int_a^b \mathbf{G}^*(x, y; t)\mathbf{v}(y, 0) \, dy + \int_a^t \int_0^s \mathbf{G}^*(x, y; t-s)\mathbf{g}(y, s) \, ds \, dy
\]

where \( \mathbf{G}^* \) satisfies \( \mathbf{G}_t^* = \mathbf{G}_{xx}^* \) with \( \mathbf{G}^*(x, y; 0) = \mathbf{I}\delta(x-y) \). Thus \( \mathbf{G}^* = \mathbf{IG}_1 \). The result follows immediately by pre-multiplying (14) by \( \exp(\mathbf{A}t) \) to obtain \( \mathbf{u} \).

3. Example

Consider two cables of length \( L \) which are coupled by conductors in such a way that the potentials \( u, v \) on the cables satisfy

\[
\begin{align*}
u_t &= u_{xx} - u + \alpha v + f \\
v_t &= v_{xx} - v + \beta u + g,
\end{align*}
\]

where \( f = f(x, t), \ g = g(x, t), \) and \( 0 < \alpha, \beta < 1 \). For definiteness we will assume Dirichlet conditions at 0 and \( L \) so that

\[
u_0 = u(L, t) = v_0 = v(L, t) = 0, \quad t \geq 0
\]

and suitable initial values of \( u \) and \( v \) are given.

The matrix \( \mathbf{A} \) is in this case

\[
\mathbf{A} = \begin{bmatrix} -1 & \alpha \\
\beta & -1 \end{bmatrix}
\]

and \( \mathbf{A}t \) has eigenvalues \(-t(1 \pm (\alpha \beta)^{1/2})\) with eigenvectors \((1 \mp (\beta/\alpha)^{1/2})^T\). Thus

\[
\mathbf{P} = \begin{bmatrix} 1 & 1 \\
-(\beta/\alpha)^{1/2} & (\alpha/\beta)^{1/2} \end{bmatrix}
\]

diagonalizes \( \mathbf{A}t \) and we find

\[
\exp(\mathbf{A}t) = \mathbf{P} \begin{bmatrix} \exp(-t(1 + (\alpha \beta)^{1/2})) & 0 \\
0 & \exp(-t(1 - (\alpha \beta)^{1/2})) \end{bmatrix} \mathbf{P}^{-1}
\]

\[
= \exp(-t) \begin{bmatrix} \cosh((\alpha \beta)^{1/2}t) & \left(\frac{\alpha}{\beta}\right)^{1/2} \sinh((\alpha \beta)^{1/2}t) \\
\left(\frac{\beta}{\alpha}\right)^{1/2} \sinh((\alpha \beta)^{1/2}t) & \cosh((\alpha \beta)^{1/2}t) \end{bmatrix}
\]

Using (8) and (10) we obtain the general solution of the coupled linear system (15)
\[ (u(x, t), v(x, t)) = \exp(-t) \begin{bmatrix} \cosh((x\beta)^{1/2}t) & \left(\frac{\alpha}{\beta}\right)^{1/2} \sinh((x\beta)^{1/2}t) \\ \left(\frac{\beta}{\alpha}\right)^{1/2} \sinh((x\beta)^{1/2}t) & \cosh((x\beta)^{1/2}t) \end{bmatrix} \]

\[ \times \left\{ \int_0^L \left( G_1(x, y; t)u(y, 0) \right) dy + \int_0^L \int_0^t \exp(s) \right\} \]

\[ \times \begin{bmatrix} \cosh((x\beta)^{1/2}s) & -\left(\frac{\alpha}{\beta}\right)^{1/2} \sinh((x\beta)^{1/2}s) \\ -\left(\frac{\beta}{\alpha}\right)^{1/2} \sinh((x\beta)^{1/2}s) & \cosh((x\beta)^{1/2}s) \end{bmatrix} \]

\[ \times \begin{bmatrix} \int G_1(x, y; t-s) f(y, s) ds \\ \int G_1(x, y; t-s) g(y, s) dy \end{bmatrix} \]

where one representation for \( G_1 \) is

\[ G_1(x, y; t) = \frac{2H(t)}{L} \sum_{n=1}^{\infty} \phi_n(x) \phi_n(y) \exp(-\mu_n^2 t) \]

with \( H(t) \) as the unit step at \( t = 0 \), \( \phi_n(x) = \sin(n\pi x/L) \) and \( \mu_n = n\pi/L \).

Finally, we point out two things. Firstly, if \( A \) is a pair of complex conjugate eigenvalues \( \lambda \pm \imath \mu \) (i.e. with non-zero imaginary part) then \( G \) will involve \( \cos(\mu t) \) and \( \sin(\mu t) \). In fact if the system of ordinary differential equations \( du/dt = Au \) has an oscillatory solution, then \( G \) will contain oscillatory components and solutions of \( u_t = u_{xx} + Au + f \) will display oscillatory behaviour. Secondly, when \( D \neq I \) in (1) it is not such a simple matter to find the Green’s function matrix as it is in the Corollary. The existence of such a Green’s function, however, follows by taking Laplace transforms and obtaining a linear system of coupled ordinary differential equations.

**REFERENCES**


