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Use of Green's function matrices for systems of diffusion equations

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The usefulness of Green's function matrices for multi-dimensional diffusion equations is indicated. A general result is obtained and an application to two coupled cable equations is made.

1. Introduction

Green's functions have provided a useful method for solution of many partial differential equations (Morse and Feshbach 1953, Roach 1970). Recently there has been much interest in multi-component reaction-diffusion equations (Aronson and Weinberger 1978, Scott and Luzader 1979, Smoller 1983, Tuckwell 1981).

A useful starting point for analysing non-linear systems of reaction-diffusion equations is found through linear systems obtained by expanding the non-linear terms about some critical point. It is the purpose of this paper to show that Green's function matrices are as useful for multi-component systems as scalar Green's functions are for one-dimensional problems.

2. Green's function matrices

We focus attention on multi-dimensional diffusion equations but bear in mind that applications can also be made to elliptic and hyperbolic equations. In particular, let $\mathbf{u} = \mathbf{u}(x, t)$ be an $n \times 1$ vector, $\mathbf{D}$ an $n \times n$ constant diagonal matrix with positive elements D_i, for $i = 1, \ldots, n$, $\mathbf{A}$ an $n \times n$ constant matrix, and $\mathbf{f} = \mathbf{f}(x, t)$ an $n \times 1$ vector. Letting subscripts x and t denote partial derivatives with respect to these variables, we are interested in finding solutions of the system of n equations

$$\mathbf{u}_t = \mathbf{D}\mathbf{u}_{xx} + \mathbf{A}\mathbf{u} + \mathbf{f}, \quad a < x < b, \quad t > 0 \tag{1}$$

with initial data

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad a \leqslant x \leqslant b \tag{2}$$

and homogeneous linear boundary conditions of the form

$$\left.\begin{aligned} \alpha_1 \mathbf{u}(a, t) + \beta_1 \mathbf{u}_x(a, t) = \mathbf{0}, \quad t \geqslant 0 \\ \alpha_2 \mathbf{u}(b, t) + \beta_2 \mathbf{u}_x(b, t) = \mathbf{0}, \quad t \geqslant 0 \end{aligned}\right\} \tag{3}$$

Let $G_1(x, y; t)$ be the Green's function for the scalar heat equation, so that G_1 satisfies

$$G_t = G_{xx}, \quad a < x < b, \quad t > 0 \tag{4}$$

with $G(x, y; 0) = \delta(x - y)$ and suitable boundary conditions at $x = a$ and $x = b$. Then it is well-known that the solution of

$$u_t = u_{xx} + h(x, t), \quad a < x < b, \quad t > 0 \tag{5}$$

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is

$$u(x,t)=\int_a^b G_1(x,y;t)u(y,0)\,dy+\int_a^b\int_0^t G_1(x,y;t-s)h(y,s)\,ds\,dy \tag{6}$$

Here it is assumed that h is such that a unique solution of (5) exists, and in the following we assume that $\mathbf{f}$ is such that (1)–(3) have a unique solution. The extension of (6) to represent the solution of (1)–(3) is then as follows.

Theorem

Let $\mathbf{G}(x, y; t)$ be the solution of

$$\mathbf{G}_t=\mathbf{D}\mathbf{G}_{xx}+\mathbf{A}\mathbf{G},\quad a<x<b,\quad t>0$$

with $\mathbf{G}(x, y; 0)=\mathbf{I}\delta(x-y)$, where $\mathbf{I}$ is the $n\times n$ identity matrix, and

$$\left.\begin{aligned}\alpha_1\mathbf{G}(a,y;t)+\beta_1\mathbf{G}_x(a,y;t)=\mathbf{0}\\ \alpha_2\mathbf{G}(b,y;t)+\beta_2\mathbf{G}_x(b,y;t)=\mathbf{0}\end{aligned}\right\} \tag{7}$$

Then the solution of (1)–(3) is given by

$$\mathbf{u}(x,t)=\int_a^b \mathbf{G}(x,y;t)\mathbf{u}_0(y)\,dy+\int_a^b\int_0^t \mathbf{G}(x,y;t-s)\mathbf{f}(y,s)\,ds\,dy \tag{8}$$

Proof

The proof proceeds as in the scalar case, that is, one checks that $\mathbf{u}(x, t)$ given by (8) satisfies:

(*a*) the differential equations (1);

(*b*) the initial condition (2); and

(*c*) the boundary conditions (3).

These are routine calculations and need not be given in detail. □

When the diffusion coefficients in each of the component equations are equal it is a simple matter to find the matrix $\mathbf{G}(x, y; t)$ which we call the Green's function matrix.

Corollary

If

$$\mathbf{D}=\mathbf{I} \tag{9}$$

then the Green's function matrix is given by

$$\mathbf{G}(x,y;t)=\exp(\mathbf{A}t)\,G_1(x,y;t) \tag{10}$$

where G_1 is the Green's function for the scalar heat equation as in (4), that satisfies the boundary conditions

$$\left.\begin{aligned}\alpha_1 G_1(a,y;t)+\beta_1 G_{1,x}(a,y;t)=0\\ \alpha_2 G_1(b,y;t)+\beta_2 G_{1,x}(b,y;t)=0\end{aligned}\right\} \tag{11}$$

Proof

When $\mathbf{D} = \mathbf{I}$ the substitution

$$\mathbf{v}(x, t) = \exp(-\mathbf{A}t)\mathbf{u}(x, t) \tag{12}$$

gives the uncoupled system

$$\mathbf{v}_t = \mathbf{v}_{xx} + \mathbf{g}(x, t) \tag{13}$$

where $\mathbf{g} = \exp(-\mathbf{A}t)\mathbf{f}$. It is clear from the scalar case that the solution of (13) is

$$\mathbf{v}(x, t) = \int_a^b \mathbf{G}^*(x, y; t)\mathbf{v}(y, 0)\, dy + \int_a^b \int_0^t \mathbf{G}^*(x, y; t-s)\mathbf{g}(y, s)\, ds\, dy \tag{14}$$

where $\mathbf{G}^*$ satisfies $\mathbf{G}^*_t = \mathbf{G}^*_{xx}$ with $\mathbf{G}^*(x, y; 0) = \mathbf{I}\delta(x-y)$. Thus $\mathbf{G}^* = \mathbf{I}G_1$. The result follows immediately by pre-multiplying (14) by $\exp(\mathbf{A}t)$ to obtain $\mathbf{u}$.

3. Example

Consider two cables of length L which are coupled by conductors in such a way that the potentials u, v on the cables satisfy

$$\left.\begin{aligned} u_t &= u_{xx} - u + \alpha v + f \\ v_t &= v_{xx} - v + \beta u + g, \quad 0 < x < L, \quad t > 0 \end{aligned}\right\} \tag{15}$$

where $f = f(x, t)$, $g = g(x, t)$, and $0 < \alpha, \beta < 1$. For definiteness we will assume Dirichlet conditions at 0 and L so that

$$u(0, t) = u(L, t) = v(0, t) = v(L, t) = 0, \quad t \geqslant 0$$

and suitable initial values of u and v are given.

The matrix $\mathbf{A}$ is in this case

$$\mathbf{A} = \begin{bmatrix} -1 & \alpha \\ \beta & -1 \end{bmatrix}$$

and $\mathbf{A}t$ has eigenvalues $-t(1 \pm (\alpha\beta)^{1/2})$ with eigenvectors $(1 \mp (\beta/\alpha)^{1/2})^{\mathrm{T}}$. Thus

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -\left(\dfrac{\beta}{\alpha}\right)^{1/2} & \left(\dfrac{\beta}{\alpha}\right)^{1/2} \end{bmatrix}$$

diagonalizes $\mathbf{A}t$ and we find

$$\begin{aligned} \exp(\mathbf{A}t) &= \mathbf{P}\begin{bmatrix} \exp(-t(1+(\alpha\beta)^{1/2})) & 0 \\ 0 & \exp(-t(1-(\alpha\beta)^{1/2})) \end{bmatrix}\mathbf{P}^{-1} \\ &= \exp(-t)\begin{bmatrix} \cosh((\alpha\beta)^{1/2}t) & \left(\dfrac{\alpha}{\beta}\right)^{1/2} \sinh((\alpha\beta)^{1/2}t) \\ \left(\dfrac{\beta}{\alpha}\right)^{1/2} \sinh((\alpha\beta)^{1/2}t) & \cosh((\alpha\beta)^{1/2}t) \end{bmatrix} \end{aligned}$$

Using (8) and (10) we obtain the general solution of the coupled linear system (15)

$$
\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \exp(-t) \begin{bmatrix} \cosh((\alpha\beta)^{1/2}t) & \left(\dfrac{\alpha}{\beta}\right)^{1/2} \sinh((\alpha\beta)^{1/2}t) \\ \left(\dfrac{\beta}{\alpha}\right)^{1/2} \sinh((\alpha\beta)^{1/2}t) & \cosh((\alpha\beta)^{1/2}t) \end{bmatrix}
$$

$$
\times \left\{ \int_0^L \begin{pmatrix} G_1(x,y;t)u(y,0) \\ G_1(x,y;t)v(y,0) \end{pmatrix} dy + \int_0^L \int_0^t \exp(s) \right.
$$

$$
\times \begin{bmatrix} \cosh((\alpha\beta)^{1/2}s) & -\left(\dfrac{\alpha}{\beta}\right)^{1/2} \sinh((\alpha\beta)^{1/2}s) \\ -\left(\dfrac{\beta}{\alpha}\right)^{1/2} \sinh((\alpha\beta)^{1/2}s) & \cosh((\alpha\beta)^{1/2}s) \end{bmatrix}
$$

$$
\left. \times \begin{pmatrix} G_1(x,y;t-s)f(y,s) \\ G_1(x,y;t-s)g(y,s) \end{pmatrix} ds\, dy \right\}
$$

where one representation for G_1 is

$$
G_1(x,y;t) = \frac{2H(t)}{L} \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y) \exp(-\mu_n^2 t)
$$

with $H(t)$ as the unit step at $t=0$, $\phi_n(x) = \sin(n\pi x/L)$ and $\mu_n = n\pi/L$.

Finally, we point out two things. Firstly, if $\mathbf{A}t$ has a pair of complex conjugate eigenvalues $\lambda \pm i\mu$ (i.e. with non-zero imaginary part) then $\mathbf{G}$ will involve $\cos(\mu t)$ and $\sin(\mu t)$. In fact if the system of ordinary differential equations $d\mathbf{u}/dt = \mathbf{Au}$ has an oscillatory solution, then $\mathbf{G}$ will contain oscillatory components and solutions of $\mathbf{u}_t = \mathbf{u}_{xx} + \mathbf{Au} + \mathbf{f}$ will display oscillatory behaviour. Secondly, when $\mathbf{D} \neq \mathbf{I}$ in (1) it is not such a simple matter to find the Green's function matrix as it is in the Corollary. The existence of such a Green's function, however, follows by taking Laplace transforms and obtaining a linear system of coupled ordinary differential equations.

References

Aronson, D., and Weinberger, H., 1978, *Adv. Math.*, **30**, 33.

Morse, P. M., and Feshbach, H., 1953, *Methods of Theoretical Physics*, Pts. I and II (New York: McGraw-Hill).

Roach, G. F., 1970, *Green's Functions* (London: Van Nostrand).

Scott, A. C., and Luzader, S. D., 1979, *Phys. Scripta.*, **20**, 495.

Smoller, J., 1983, *Shock Waves and Reaction-Diffusion Equations* (New York: Springer-Verlag).

Tuckwell, H. C., 1981, *Int. J. Neurosci.*, **12**, 95.