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MATRIX METHODS FOR PREDICTING AUSTRALIA'S POPULATION

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The population of Australia obtained from the 1976 census was 13 991 200. From census data, the Australian Bureau of Statistics has predicted that the total number of people in Australia will be 15 595 600 by 1986 and will rise to 18 867 300 by the year 2006. Figure 1 shows the populations up to and including 1976 obtained from census data, as well as future predictions. These future predictions, or *projections* as they are called, were obtained by what is known as the *component method*, in which the dynamics of population growth are broken up into the components of birth, death and migration.

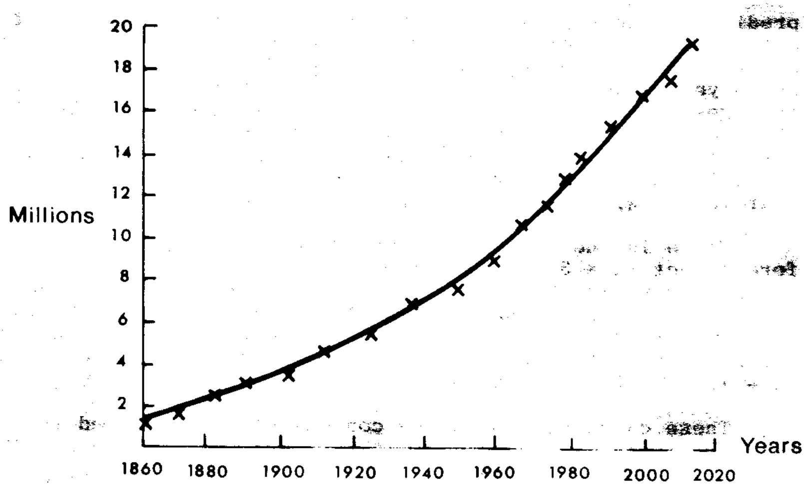

Figure 1. The population of Australia from 1860 onwards. The figures prior to 1976 are from Census data: later figures are predictions using the component method.

We shall explain in this article the basic mathematical ideas behind the component method. In principle these were first outlined in a general form by P.H. Leslie in an article for the journal *Biometrika* in 1945. Henceforth we focus attention on the females of the population, because they are the members that give birth to new individuals.

We must first subdivide the population into groups of various ages. This is because females of various ages have children at different rates and also, older females are usually more likely to die than younger ones. The rate at which females in a certain age group give birth is called an *age-specific fertility rate*. The rate at which they die is called an *age-specific mortality rate*.

Let us illustrate. In 1976, amongst 1000 Australian women between the ages of 20 and 24, there were on average 129 children born. This gives an age-specific fertility rate of ·129. By contrast, the age-specific fertility rate for women in the 35-39 year age group was ·024. Similarly, in 1976, amongst 2000 Australian women between the ages of 20 and 24 years there was on average only one death, yielding an age-specific mortality rate of ·0005. On the other hand, women between 70 and 74 years old had an age-specific mortality rate of ·03, representing a sixty-fold increase.

Births and deaths are not the only causes of changes in population size. In developing countries such as Australia, *migration* also plays a significant role. In 1976, for example, there was an increase in Australia's population of about 26 000 due to immigration, which is non-negligible in comparison with the 200 000 or so births.

Let us see how to formulate mathematically the problem of predicting the population in the future from these kinds of data. For simplicity we shall suppose there are only three age groups: up to one year; between 1 and 2 years; and between 2 and 3 years (we assume no one lives past three years). Let the age-specific fertility rates for these age groups be ·1, ·2, and ·1 respectively, and the corresponding age-specific mortality rates be ·05, ·1, and 1·0. Suppose that there are initially 300, 200, and 100 in the respective age groups. One year later, what is the age distribution of the population?

Those in the 0 - 1 age group are all from births; we therefore expect $\cdot 1 \times 300 + \cdot 2 \times 200 + \cdot 1 \times 100 = 80$ in this age group. Those in the 1 - 2 age group are survivors from the original 0 - 1 age group; we expect $(1 - \cdot 05) \times 300 = 285$ in the 1 - 2 age group. Similarly, those in the 2 - 3 year age group are survivors from the initial 1 - 2 year age group; we expect $(1 - \cdot 1) \times 200 = 180$ in the 2 - 3 year age group.

These calculations may be conveniently summarized in matrix notation as follows. Let N_0 and N_1 be the 3×1 column vectors giving the age distributions for the three age groups at times $t = 0$ and $t = 1$ respectively; for example, here

$N_0 = \begin{bmatrix} 300 \\ 200 \\ 100 \end{bmatrix}$ and $N_1 = \begin{bmatrix} 80 \\ 285 \\ 180 \end{bmatrix}$. Let M denote the 3×3 matrix, the first row of which consists of the age-specific fertility rates, the second row of which contains all zeros except for the first element, which is the age-specific survival rate (that is, one minus the age-specific mortality rate) for the first age group, and the third row of which contains all zeros except for the second element, which is the age-specific survival rate for the second age group. Thus for our example,

$$M = \begin{bmatrix} \cdot 1 & \cdot 2 & \cdot 1 \\ \cdot 95 & 0 & 0 \\ 0 & \cdot 9 & 0 \end{bmatrix} .$$

Now, our system of three linear equations,

$$\begin{aligned} \cdot 1 \times 300 + \cdot 2 \times 200 + \cdot 1 \times 100 &= 80 \\ (1 - \cdot 05) \times 300 + 0 \times 200 + 0 \times 100 &= 285 \\ 0 \times 300 + (1 - \cdot 1) \times 200 + 0 \times 100 &= 180 , \end{aligned}$$

can, by the rules of matrix multiplication, be written in mathematical shorthand notation as

$$M\ N_0 = N_1 .$$

Using the same rule, the vector N_2, whose elements are the projected numbers in each age group at time $t = 2$, can be found by multiplying the vector N_1 by the matrix M. That is,

$$M\ N_1 = \begin{bmatrix} \cdot 1 & \cdot 2 & \cdot 1 \\ \cdot 95 & 0 & 0 \\ 0 & \cdot 9 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 285 \\ 180 \end{bmatrix} = \begin{bmatrix} 88 \\ 76 \\ 256 \cdot 5 \end{bmatrix} = N_2 .$$

This procedure may be repeated indefinitely to predict the numbers in each age group in future generations so long as fertility rates and mortality rates remain fixed.

We call the matrix M a *Leslie matrix* after P.H. Leslie. Different Leslie matrices and different initial age distributions give rise to different patterns of long term population changes. An interesting example provided by Leslie occurs if

$$M = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} .$$

Here we have a population which lives for only three years and which reproduces only in the third year of life. Let

$$N_0 = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} .$$

After one year, we expect an age distribution given by

$$M\,N_0 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 500 \\ 333\frac{1}{3} \end{bmatrix} = N_1\ ;$$

after two years, we expect an age distribution given by

$$M\,N_1 = M^2 N_0 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 6000 \\ 500 \\ 333\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2000 \\ 3000 \\ 166\frac{2}{3} \end{bmatrix} = N_2\ ;$$

and after three years, we expect an age distribution given by .

$$M\,N_2 = M^2\,N_1 = M^3\,N_0 = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 2000 \\ 3000 \\ 166\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = N_3.$$

Note that $N_3 = N_0$. That is, the age distribution after three years is the same as it was at the start! You should be able to see that in fact $N_6 = N_3$, $N_9 = N_6$, and so forth, so the population values repeat every three years.

Now suppose that the Leslie matrix M is given by

$$M = \begin{bmatrix} 0 & 1 & 3 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix},$$

and that the initial population distribution is again given by

$$N_0 = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

We shall not go through the calculations here, but we may show that, with these values for M and N_0, the initial population will tend toward a total population of 4000 distributed in the ratio of (6:3:1) for all subsequent generations; this age distribution will be achieved after approximately 23 generations. This is an example of a population that tends toward what is known as a *stable distribution*, that is, a population distribution in which the ratios of numbers in different age groups remain constant through generations.

As we had mentioned previously, the Australian Bureau of Statistics has made projections of Australia's population using the component method. Naturally, the calculations involved are much more complicated than the ones that we have discussed, but the ideas behind them are the same. The projections are quite useful for government and industry as a means of predicting future demand for goods and services. For example, public policy planners use population projections to allocate funds for the construction of schools, roads, and other vital services that a future population may require. Population projection can be a hazardous undertaking, however, in that we are applying assumptions in the present about future trends of fertility, mortality and the characteristics of overseas migration. If our assumptions are not borne out with time, then our projections can be grossly in error.

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