# Shift of equilibria by noise

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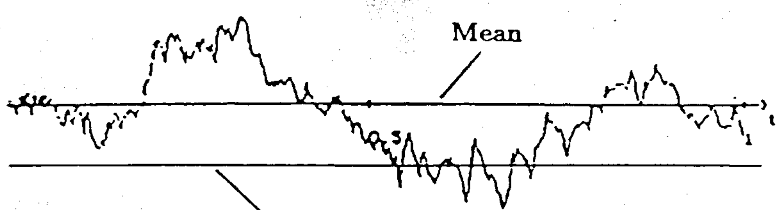
**The effects of unpredictable fluctuating influences in physical systems have been studied since early this century. In linear systems the noise does not affect mean values at equilibrium points. In nonlinear systems, however, we find that the mean is displaced from from an equilibrium point. This effect, which is general and is not due simply to asymmetries, means that measurements of an equilibrium point must be corrected. A formula for the correction factor has been found.**

Noise of small amplitude is very often present in recordings made in many areas of science. Current and voltage recordings from nerve cells [1,2] provide one example. Fluctuations are also found in ion concentration measurements in brain structures [3] and in electroencephalographic recordings [4]. They have recently been included in models of tectonic movement[5]. We have found that the average of a dynamical quantity as it fluctuates randomly about an equilibrium point in a nonlinear system is usually shifted from that equilibrium point . This shift, whose direction depends on local properties , is illustrated in the Figure , and makes incorrect an estimate of the equilibrium point obtained by averaging .




Equilibrium value

Fluctuation phenomena that have been widely studied since early this century [6] , include the influence of a fluctuating force on the motion of a particle. If X(t) is the speed of such a particle at time t , then Newton's second law gives dX/dt = -X + w(t), where it is assumed that the particle has unit mass and $w(t), t \geq 0$, is called white noise [7] . The mean value of this noise process is zero. The random quantity X(t),$t \geq 0$ is called an Ornstein-Uhlenbeck process [8]. In the absence of the random forces the particle eventually returns exponentially in time from any initial value to the origin X = 0, and we say that the origin is a stable equilibrium point. When the random forces are present, the expected value of X(t) also returns to the equilibrium point as time increases.

However, in many situations of interest the non-random part of the force may be nonlinear and there has been much interest recently in such nonlinear systems when there is noise present [9,10,11] . This leads us to consider the equation $dX/dt = f(X) + \epsilon w$, where $\epsilon$ is a small number and f is a nonlinear function. We investigate the behaviour of the mean of X(t) when fluctuations occur about an equilibrium point: i.e., a point $x_0$ such that $f(x_0) = 0$ . It is assumed that $x_0$ is stable which is the case if the first derivative of f is negative at $x_0$ .

Because $\epsilon$ and hence the random perturbation is small , we express X as $X(t) = x_0 + \sum_{k=1}^{\infty} \epsilon^k X_k$. When we substitute this expression in the differential equation we get equations for each of the terms $X_1$, $X_2$, ... . The expected value of $X_1(t)$ is zero.

Putting $f_0'' = f''(x_0)$ , we find the following expression for the expected or mean value of X(t):

$$E(X(t)) \simeq x_0 + \frac{\epsilon^2 M(t) f_0'' e^{f_0' t}}{2},$$

where M(t) stands for the the mean value of an integral which is always positive and terms of order three and higher in $\epsilon$ have been dropped. It is apparent therefore that the mean of X(t) is above $x_0$ if $f_0''$ is positive , whereas the mean of X(t) is below $x_0$ if $f_0''$ is negative.

One can determine M(t) exactly directly by using the correlation function of white noise. With this value of M(t) the mean of X(t) is approximately

$$E(X(t)) \simeq x_0 + \frac{\epsilon^2 f_0'' e^{-\alpha t}}{2\alpha^2}[cosh(\alpha t) - 1],$$

where $\alpha$ is the absolute value of $f_0'$. Furthermore, as $t \to \infty$, E(X(t)) approaches the approximate steady state value $E(X(t)) \to x_0 + \frac{\epsilon^2 f_0''}{4\alpha^2}$. Thus the mean does not approach the equilibrium value as in the linear case (Ornstein-Uhlenbeck process).

One sees therefore that for all t , and in particular in the steady state ($t = \infty$) that the mean of the dynamical quantity X(t) is , if $f_0'' \neq 0$, displaced from the equilibrium point $x_o$. This occurs despite the fact that the added noise itself has zero mean and implies that any attempt to measure $x_o$ by averaging a record of X(t) will lead to an erroneous result. The true equilibrium value must be estimated in the steady state by using the appropriate correction factor.

One example from statistical physics is the stochastic Landau equation [12] , $dX/dt = (\lambda X - X^3) + \epsilon w$, where $\lambda$ is a constant. Here $f(x) = \lambda x - x^3$ and there are two stable equilibrium points, $x_0 = \pm\sqrt{\lambda}$. Differentation yields $f_0' = -2\lambda$ and $f_0'' = \mp 6\sqrt{\lambda}$ . Substituting these values above shows that at the upper equilibrium point the mean is moved downwards whereas at the lower equilibrium point, the mean is moved upwards .

## References and Notes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