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THE BRAIN AS AN ANALOG COMPUTER

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Dedicated in friendship and gratitude to Marian Pour-El.

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Abstract

We draw attention to some of the ways the brain acts as an analog computer, rather than as a digital computer. The outputs of analog computers are, by the Shannon–Pour-El Thesis, known to be exactly the solutions of algebraic differential equations in each of the individual variables. Hence numerous mathematical theorems about such solutions can be interpreted in the context of the functioning of the central nervous system.

1. Introduction

One model of the functioning of the brain is that it is an assemblage of interconnected neurons, each of which may fire an impulse (a sort of delta function) down its axon to the dendrites of connected neurons, which then fire spikes of their own, and so on. This picture is reminiscent of a digital computer, and we shall call it "the digital model", although "binary model" would be more appropriate. An early mathematical proponent of this digital approach was John von Neumann (see [NEU]), and it has had a wide influence, especially within the Artificial Intelligence school.

In this paper, we draw attention to some of the ways the brain, or parts of it, acts as an *analog* computer.

What is an analog computer? This is a broad term that can range from a wind tunnel or the David Taylor Model Basin, to the mechanical differential analyzer of Bush at M.I.T. in the 1940s, to the electrical analog computers of later years, and presently to so called analog filters (which are essentially miniaturized analog computers). On this

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broad a scale, perhaps they should be called "continuous computers", following Goldstine (see [GOL]).

But rather than discuss the hardware of analog computers, let us concentrate on the "general-purpose analog computer" (GPAC), which is really a mathematical concept, even though it is currently most often implemented with electronic circuits containing various devices — notably the operational amplifier as an integrator. The mathematical side of the GPAC has been admirably described in [SHA] and [POE], in terms of certain systems of linked non-linear algebraic differential equations (ADE's) of the first order, which we will describe shortly.

Since awareness seems to be growing that some of the brain's function is analog in nature, perhaps the main contribution of this paper is to focus attention on the GPAC, which is an unambiguous mathematical concept with a precise function, and about which there are precise mathematical theorems. For "analog" is something of a catch-word that is sometimes loosely used, in contexts ranging from wristwatches to plumbing.

In a GPAC, we have the time variable t, which it is also convenient to write as $y_1 = y_1(t)$. Then there are finitely many dependent variables $y_2 = y_2(t), \ldots, y_n = y_n(t)$, that are constrained to satisfy, for $i = 2, 3, \ldots, n$, the system

$$\frac{dy_i}{dt} = \sum_{j,k=1}^{n} c_{ijk}\, y_j \frac{dy_k}{dt},$$

where the c_{ijk} are real constants.

It is edifying to think of a GPAC as a system of "black-boxes" of four elementary kinds, without any inquiry into the insides of the boxes, which may vary from context to context. These boxes are hooked up with lots of feedback. The first kind of box is a "constant box" that produces any constant voltage desired. Then there is an adder that has inputs u and v and produces the output $u + v$. Here u and v are $u(t)$ and $v(t)$, which are functions of real time t. Then there is the multiplier that has inputs u and v and produces the output $u \cdot v$. Finally, and most importantly, there is the *integrator* that has inputs u and v and produces the output $\int_0^t u(s)\,dv(s) + C$. Here, the integral is the Riemann-Stieltjes integral that may be thought of here as $\int_0^t u(s)\,v'(s)\,ds$, where $v'(s) = dv/ds$. The constant C in the above formula is referred to as the "initial setting" of the integrator. Let us mention first, that by concatenation of adders, there is no difficulty in introducing an adder that has inputs $u_1, u_2, \ldots, u_m$ for any whole number m, and output $u_1 + u_2 + \ldots + u_m$. Also, the formula $uv = \int u\,dv + \int v\,du$ of integration by parts shows that we may dispense with multipliers if we wish, and replace each of them by two integrators and an adder. For a more detailed description of GPAC's, in particular for the natural restrictions on how the black boxes may be interconnected, see [POE]. Once the connections are made and the initial settings made, the GPAC is just permitted to run in real time. Any voltage that can be read in the circuit (i.e. any of the y_i) as a function of t is called an "output".

In our context, one wishes to identify neurons or neuron-circuits that can function as such primary black boxes. Candidates for constant-boxes and adders (that perform spatial summation) are certainly well known. And integrators have been identified and studied, at least in the context of the oculomotor system, for the past sixteen years. (See [ROB–I], [ROB–II], and [CRS] for discussions of pools of neurons that integrate premotor neural commands. The introduction to this last paper has an especially valuable historical discussion, mentioning [CKO] which provided the first physiological evidence.) One would expect other integrator mechanisms to be identified in the course of time. So it seems that at least the basic *components* for GPAC's may be found in the central nervous system.

2. The Shannon–Pour-El Thesis

For our purposes, the most important theorem about GPAC's is the Shannon-Pour-El Thesis that the outputs of GPAC's correspond exactly to differentially algebraic (DA) functions, that is, to functions $u(t)$ that satisfy an ADE in their own right, like

$$(t^2 + 5)u^2u'^3u'' - (t^3 + \pi t)uu''u'''^2 + u' + 3t - 5 = 0.$$

Equations like this may be highly nonlinear, and of very high order. However, we stress that unlike the systems we have discussed earlier, these equations involve only one dependent variable at a time.

The actual proof of the Shannon–Pour-El Thesis is divided into two parts. The first part shows how, given an equation like the last one, one proceeds by prescribed steps to wire up a GPAC that solves it. The reverse direction, given an output of a GPAC, to find an ADE that it solves, involves so called elimination theory, and the incomplete proofs in [SHA] and [POE] have only recently been emended in [LIR]. We mention that it is shown in [RUS] that certain much more complex equations, which may contain highly transcendental functions, can be reduced to ADE's (and hence to GPAC's) by a further process of elimination. Thus, one can also solve, on a GPAC, such equations as

$$ty'^{y''} + \log \sin\left(t + \arctan \frac{y + y''}{ty'^2}\right) + J_\upsilon((t+1)y) = 0.$$

We have included J_υ not merely for effect, but because this Bessel function plays a central role in vibrations of certain membranes, like the eardrum.

Now "most" of the elementary functions of mathematical analysis are differentially algebraic. Hence they can be generated by analog computer — or by the brain, from our point of view. (The one exception that an undergraduate mathematics major might encounter is the Euler gamma function,

$$\Gamma(x)=\int_0^\infty t^x e^{-t}\,\frac{dt}{t}$$

— see below for an expansion of this remark.)

These analog computable (DA) functions certainly include polynomials, like $5t^3-\pi t^2+\sqrt{2}$, rational functions, like $(t^3-5)/(t^2+1)$, and *algebraic* functions, like $\sqrt[5]{t^3-4^e}$, but many others as well. For example, exponentials like e^{5t} (which satisfies the ADE $y'=5y$). Also trigonometric functions, like sin ωt (alpha waves?) which satisfies $y'^2+\omega^2y^2-\omega^2=0$, (or $y''+\omega^2y=0$) and also the Bessel functions $J_\upsilon(t)$ which satisfy Bessel's equation (a *linear* ADE). Further, by [OST] and [MOO], any combination of DA functions, so long as it involves only sums, products, quotients, differences, compositions, and compositional inverses, is again DA. So the analog brain can produce such voltages, say, as

$$J_5(e^{e^{et}}-\sin t)+\arctan(\log\sec\sqrt{t}),$$

and so on.

So analog computers can produce a very rich class of functions. Indeed, this class is so rich that *any* continuous function, either on a closed interval $t_0\le t\le t_1$ or on the whole positive time axis $0\le t<\infty$, can be approximated with arbitrary preassigned accuracy by the output of a GPAC. Surprisingly, there exist "universal" analog computers with only a handful of integrators, that have outputs that approximate arbitrarily well any continuous function in the world. (See [RUB], [DUF], [BUC], and [BOS] for details.) For example, an equation in [DUF] with this property is

$$(*)\qquad 2y''''y'^2-5y'''y''y'+3y''^3=0,$$

so that the corresponding analog computer with only *four* integrators produces every conceivable voltage pattern, to within an arbitrarily small error. The equation in BOS], although presumably much more complicated, is considerably more deterministic than (*). However, it has the universal property only on closed and bounded ntervals of time.

Further, it is shown in [RUS] that a large class of problems, to minimize certain integrals that arise in the calculus of variations, can be solved on analog computers. Many minimization problems of elementary physics, like the tautochrone or the brachistochrone, fall under this rubric.

It must be mentioned (see [RIG], [BAK], [STA]) that there *do* exist many "transcendentally transcendental" functions which are not *exactly* the solutions of any ADE's, and are consequently not *exactly* the output of any GPAC. In particular, the function $\Gamma(x)$ (which gives a continuous extension of the factorial function via $n!=\Gamma(n+1)$) is not such an output, even though it may be arbitrarily well approximated by such outputs. Perhaps this should be considered as an essential limitation of n analog brain. But in the real physical world, with its inherent limitations on precision to begin with, this seems a moot point. We hope we have made it clear that ven small analog computers can have an extraordinarily rich repertoire of outputs.

The fundamental point of the Shannon–Pour-El Thesis is that it provides an underlying duality between GPAC's and ADE's. Every statement about analog computers has its counterpart in a statement about differentially algebraic functions, and vice versa. To this we wish to add a third facet, that if we consider the brain as an analog computer, then each statement in any one of the three disciplines of analog computing, algebraic differential equations, or neuropsychology, has its natural counterparts in the other two domains.

3. Redundancy

Let us illustrate this by describing a corollary of the famous Ritt-Raudenbush Theorem of differential algebra. This corollary asserts (see [KAP]) that given any system Σ, presumably an infinite one, of ADE's in a finite number of unknown functions $Y_1(t),\dots,Y_n(t)$, there is a *finite* subsystem Σ_f of the original system that has exactly the same solutions, no more and no less, as the original system Σ. That is, all but finitely many of the equations in Σ are redundant.

It is tempting to parallel this by saying that if we imagine an idealized brain with infinitely many neurons, operating as a gigantic analog computer (with each neuron having only finitely many predecessors in the wiring diagram), then we may throw away all but a few of the neurons, and the few remaining ones perform all the same functions as the original vast brain. This then, is a mathematical approach to redundancy in the central nervous system. (Of course, this refers to ongoing redundancy versus the redundancy of repair — the brain's recovery from injury is complex and takes time.) Here, we have approximated a very large system by an infinite system — a common type of idealization in science. What is needed to make this approach more biologically realistic is a quantitative version of the Ritt-Raudenbush theorem, that would start with a large finite system of equations, that is in some sense generic, and reduce it to a relatively small subsystem.

Let it be pointed out that the truth of the Ritt-Raudenbush theorem depends on the *smoothness* assumptions made about the solutions. If they are assumed to have derivatives of arbitrarily high order, so that they are infinitely smooth, then the theorem (and its proof in [KAP]) are valid. But it was shown in [RUB–II] that if we suppose that the solutions have derivatives only up to a certain fixed finite order, then the conclusion fails. This naturally leads to the question — "how smooth are the voltages in the brain?" On it might depend the degree of redundancy in neuronal circuits. This question is admittedly hard to formulate precisely or to devise corresponding measurements for, especially in the light of eventual quantum-theoretical phenomena. But the question of smoothness (which we have suppressed until now) is fundamental to the whole qualitative theory of analog computers (read ADE's). For example, although it is well-known that the sums, products, etc., of two infinitely smooth DA functions must again be DA, it was shown in [RUB–II] that this can fail for DA functions that have only a fixed finite number of derivatives.

In this direction, it should be mentioned that the outputs cannot be arbitrarily pathological — they must have certain smoothness properties much of the time. See [BRR] and [LIR], where it was proved that a DA function must be analytic (i.e. represented by a convergent power series — the ultimate kind of smoothness) on a dense open set — i.e. on the union of a collection of open intervals that intersects *every* open interval. In a topological sense, then, every DA function must be analytic almost everywhere. However, see also [RUB–II] where DA functions (and hence outputs of GPAC's) are produced that are no smoother than this restriction requires. Hence this restriction cannot be improved.

4. Deterministic and stochastic theories

The question of *determinism* plays an essential part, that we have suppressed until now, at least in the *mathematical* theory of analog computers. In [POE], Pour-El seems to have found the exactly appropriate definition of determinism, that of "domain-of-generation". This not only demands that the differential equations that describe the GPAC have a (locally) unique solution for the given initial conditions (i.e. the constant settings of the integrators at time zero), but that they also have a (locally) unique solution even if the initial conditions are slightly perturbed. This means essentially that minor "errors" in the hardware, or the initial settings of the integrators, or in determining time zero, do not destroy the deterministic character of the machine. It seems not to be known whether there is any algorithm (i.e. definite routine procedure), given the wiring diagram of a GPAC, for deciding whether it has a domain-of-generation, and it seems unlikely that such an algorithm exists. Whether the brain is a deterministic system, then, seems to be beyond the reach of contemporary mathematics, and may well be beyond the reach of *any* mathematics, for that matter. (See [DNL] where it is proved that certain similar problems have no algorithm.)

We stress here that we are talking about determinism of an *ideal* GPAC, and not allowing such elements as random noise into our considerations, although as Eric Jakobsson suggests, a model of a GPAC with random (say Gaussian) fluctuations in (membrane) potentials, would correspond more closely to actual measurements. However, this could introduce substantial new difficulties into the mathematics of the situation — one would have to create *stochastic* differential algebra.

5. Conclusions

It is fashionable nowadays to downgrade analog computers, largely because of their unreliability and lack of high accuracy (roughly one-tenth of one percent at best). But analog computers, besides their versatility, are extremely fast at what they do, which is solving differential equations. In principle, they act instantaneously and in real time. Further, in contrast to the situation in digital computing, the operator of an analog computer has an extremely good "feel" for what the computer is doing. Analog computers are still unrivalled when a large number of closely related differential equations must be solved. But *digital simulation* of analog computers offers some of the best of both worlds (see the excellent discussion in Chapter 8 of [OSS]). In such digital simulation, the analog adders, multipliers, and especially integrators, are replaced by digital counterparts, but from this point, the logic of the computer is *analog*. That is, it functions exactly as a GPAC except that the black-boxes happen to have digital hardware in them. It is conceivable, as Emanuel Donchin has suggested, that the central nervous system operates, in part, in a similar fashion. We mention that Rada has shown in [RAD] that a *hybrid* computer can attain unlimited "efficiency", in contrast to the theoretical limit of efficiency ≤ 1 of the digital computer.

One final fact that needs to be mentioned is that any physical computer, the brain included, has certain delays in its functioning, if only the tiny delays in electrical transmission over short distances. Delay differential equations are coming to be better understood (see [DRI] for a reasonably elementary account), but there is no counterpart yet to the extensive algebraic theory for non-delay differential equations to be found in [KOL] and elsewhere. (There *is* however an extensive theory (see [COH]) of pure *difference* equations, which are delay equations without derivatives in them.)

In summary, if one accepts, at least tentatively, the thesis that the central nervous system at least partially functions like an analog computer, then the existing literature on algebraic differential equations becomes available to extract parallels from. (This literature is considerable, but not vast.) One can then also be led, as I have been, to start with brain functioning, and pursue certain purely mathematical theorems that are suggested by the parallel with the algebraic differential equations of general-purpose analog computers. (See [RUB–III] for such a theorem, which can be proved via a Gedankenexperiment.)

We conclude by suggesting that the brain can suggestively be modelled by a hybrid computer — half digital and half analog, with some sort of interface between them.

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References

[BAK] Steven B. Bank and Robert P. Kaufman, A note on Hölder's theorem concerning the gamma function, *Math. Annalen* **232** (1978), 115–120.

[BOS] Michael Boshernitzan, Universal formulae and universal differential equations, *Ann. of Math.*, to appear.

[BRR] Andrew M. Bruckner, Melvyn Rosenfeld, and Lee A. Rubel, The Darboux property, and solutions of algebraic differential equations, *Pacific J. Math.*, **118** (1985), 295–302.

[BUC] R. C. Buck, The solutions to a smooth PDE can be dense in C(I), *J. Differential Equations* **41** (1981), no. 2, 239–244.

[CAG–I] William H. Calvin and Katherine Graubard, Styles of neuronal computation. *The Neurosciences, Fourth Study Program.* Ed. by F. O. Schmitt and F. G. Worden, pp. 513–524, M.I.T. Press 1979.

[CAG–II] ——, Presynaptic Dendrites. Implications of Spikeless Synaptic Transmission and Dendritic Geometry. *The Neurosciences, Fourth Study Program.* Ed. by F. O. Schmitt and F. G. Worden, pp 317–331, M.I.T. Press, 1979.

[CKO] B. Cohen and A. Komatsuzaki, Eye movements induced by stimulation of the pontine reticular formation: evidence for integration in oculomotor pathways, *Exp. Neurol.* **36** (1972), 101–117.

[COH] Richard M. Cohn, *Difference Algebra*, Interscience, 1965, John Wiley and Sons, New York.

[CRS] Stephen C. Cannon, David A. Robinson, and Shihab Shamma, A proposed neural network for the integrator of the oculomotor system, *Biological Cybernetics* **49** (1983), 127–136.

[DNL] J. Denef and L. Lipshitz, Power series solutions of algebraic differential equations, *Math. Ann.* **267** (1984), 213–238.

[DRI] R. D. Driver, *Ordinary and Delay Differential Equations*, Applied Mathematical Sciences series, vol. 20, Springer-Verlag, 1977.

[DUF] Richard J. Duffin, Rubel's universal differential equation, *Proc. Nat. Acad. Sci. U.S.A.* **78** (1981) no. 8 part 1, 4661–4662.

[GOL] Herman H. Goldstine, *The Computer from Pascal to von Neumann*, Princeton University Press, 1972.

[KAK] Granino A. Korn and Theresa M. Korn, *Electronic Analog Computers*, McGraw-Hill 1952.

[KAP] Irving Kaplansky, *An Introduction to Differential Algebra*, Paris 1957.

[KOL] E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, 1973.

[LIR] Leonard Lipshitz and Lee A. Rubel, A differentially algebraic replacement theorem, and analog computability, *Proc. Amer. Math. Soc.*, to appear.

[MOO] E. H. Moore, Concerning transcendentally transcendental functions, *Math. Annalen* **48** (1896–97), 49–94.

[NEU] John von Neumann, The General and Logical Theory of Automata. *Cerebral Mechanisms in Behaviour*, The Hixon Symposium, p. 1–31, John Wiley and Sons, 1951.

[OSS] R. J. Ord-Smith and J. Stephenson, *Computer Simulation of Continuous Systems*, Cambridge Computer Science Texts 3, Cambridge University Press, Cambridge, 1975.

[OST] Alexander Ostrowski, Über Dirichletsche Reihen und algebraische Differentialgleichungen, *Math. Zeitschrift* **8** (1920), 241–298.

[POE] Marian Pour-El, Abstract computability and its relation to the general purpose analog computer, etc., *Trans. Amer. Math. Soc.* **199** (1974), 1–29.

[RAD] Roy Rada, Efficiency N^2 in hybrid N-processor, preprint, Wayne State University, 1983.

[RIG] J. F. Ritt and E. Gourin, An assemblage-theoretic proof of the existence of transcendentally transcendental functions, *Bull. Amer. Math. Soc.* **33** (1927), 182–184.

[ROB] Alan Roberts and M. H. Bush, eds, *Neurones Without Impulses*, Cambridge University Press (1981).

[ROB–I] D. A. Robinson, Eye movement control in primates, *Science* **161** (1968), 1219–1224.

[ROB–II] ——, Models of oculomotor neural organization. In: *The control of eye movements*, P. Bach-y-Rita, C. C. Collins, and J. E. Hyde (eds), pp. 519–538, Academic Press, New York, 1971.

[RUB–I] Lee A. Rubel, A universal differential equation, *Bull. Amer. Math. Soc.* (new series) **4** (1981), 345–349.

[RUB–II] ——, Solutions of algebraic differential equations, *J. Differential Equations* **49** (1983), 441–452.

[RUB–III] ——, An elimination theorem for systems of algebraic differential equations, *Houston J. Math.* **8** (1982), 289–295.

[RUS] Lee A. Rubel and Michael F. Singer, A differentially algebraic elimination theorem with applications to analog computability in the calculus of variations, *Proc. Amer. Math. Soc.*, **94** (1985), 653–658.

[SHA] Claude Shannon, Mathematical theory of the differential analyzer, *J. Math. and Physics* **20** (1941), 337–354.

[STA] V. E. E. Stadigh, Ein Staz ueber Funktionen die algebraische Differentialgleichungen befriedigen, etc., *Thesis*, Helsinki 1902.