MULTIPLE PULSE PROPAGATION IN A FITZHUGH-NAGUMO NERVE MODEL

STEPHEN LUZADER
Physics Discipline, University of Wisconsin-Parkside
P.O. Box 2000, Kenosha, Wisconsin 53141, U.S.A.

(Received November 8, 1983)

Abstract
A constant stimulus current applied to a FitzHugh-Nagumo nerve model will produce repetitive pulses over a range of current amplitudes. Integrating the model equations for a membrane patch (no propagation) and injecting current into an electronic analogue show that once a threshold for multiple pulses is exceeded, pulse period decreases with increasing current amplitude until the period reaches a minimum, and then the period increases until multiple pulse production ceases at some maximum input current. If a constant current is injected into one end of a neuristor constructed from several stages of "patch" circuits, propagating pulse trains are generated over a range of input currents. The period of these pulse trains exhibits a complex dependence on current amplitude, with abrupt increases in period occurring for some currents. Near such transition amplitudes, patterned pulse trains are seen.

1. Introduction
The FitzHugh-Nagumo model of nerve pulse propagation can be written as a nonlinear diffusion system (Scott, 1977; Luzader, 1979):

\[ \begin{align*}
V_t &= V_{xx} - F(V) - W + I(x, t) \\
W_t &= \epsilon V - bW
\end{align*} \]

where \( V \) is a dimensionless variable representing nerve membrane potential, \( F(V) \) is a nonlinear function representing the current-voltage characteristic of nerve membrane, \( W \) is an auxiliary "recovery" variable, \( I(x, t) \) represents an externally applied current, and \( \epsilon \) and \( b \) are small positive parameters.

A simplified version of Eqs (1) can be used to represent a "patch" of membrane or a space-clamp experiment in which no propagation occurs. Setting \( V_{xx} = 0 \) and suppressing \( x \) gives

\[ \begin{align*}
V' &= - F(V) - W + I(t) \\
W' &= \epsilon V - bW
\end{align*} \]
Here the prime denotes differentiation with respect to \( t \). The function \( F(V) \) is often taken to have one of two particular forms, a cubic and a piecewise linear function:

\[
\begin{align*}
F(V) &= AV(V - \theta)(V - 1), \\
F(V) &= G_1V, \ V < \theta \\
&= G_2(V - 1), \ V \geq \theta
\end{align*}
\]

The cubic form resembles rather closely the actual current-voltage relation for nerve membrane, while the piecewise linear form has the advantage of allowing many solutions to Eqs (1) to be found in closed form (Scott, 1977; Luzader, 1979; Rinzel and Keller, 1973). The parameter \( \theta \) is the threshold for impulse generation.

It is found that for \( I(t) = I_0 = \text{constant} \), \( I_0 \) is a bifurcation parameter for Eqs 2 (Troy, 1976). \( I_0 \) must exceed some minimum threshold value to have solutions for \( V(t) \) corresponding to single pulses. As \( I_0 \) is increased to another threshold value, periodic solutions for \( V(t) \) exist. At a still higher value of \( I_0 \), periodic solutions are no longer found.

If the stimulus current \( I(x,t) \) is suppressed in Eqs (1) and the requirement that \( V \) be equal to the threshold value is imposed as a boundary condition, travelling wave solutions of the form \( V(x,t) = V(x - ut) = V(\xi) \) can be found. Under this condition, Eqs (1) reduce to ordinary differential equations:

\[
\begin{align*}
-uV'' &= V' - F(V) - W \\
-uw'' &= \xi V - bW
\end{align*}
\]

Here the prime denotes differentiation with respect to \( \xi \). Eqs (4) have travelling solitary wave solutions for certain ranges of the parameter values (Scott, 1977; Luzader, 1979; Rinzel and Keller, 1973), and the existence of a variety of multiple pulse travelling wave solutions has been shown. Periodic solutions have been calculated using the piecewise linear form of \( F(V) \) (Hastings, 1974, 1976) and the existence of finite and nonperiodic infinite pulse trains has been demonstrated (Ferone, 1981). In addition to these travelling wave solutions, numerical integration of Eqs (1) for \( I(0,t) = \text{constant} \) have yielded propagating solitary pulses and pulse trains (Rinzel, 1977; Manoranjan, 1982).

**Numerical Studies of Eqs (2)**

Eqs (2) were integrated numerically using the cubic form of \( F(V) \) with \( A = 6, \theta = 0.3, \varepsilon = 0.1, b = 0.05 \) for various values of \( I_0 \). Some typical solutions are shown in Fig. 1. For \( I_0 < 0.17 \), no pulses were produced. For \( 0.17 < I_0 < 0.4 \), the solution is a single pulse followed by either an elevation in potential or some small damped oscillations. For \( 0.4 < I_0 < 0.94 \), large amplitude oscillations in \( V \) were seen, while still

![Fig. 1. Results of integrating Eqs (2) for different values of the dimensionless input current \( I_0 \), the amplitude of which is given in each graph. For the threshold value \( I_0 = 0.17 \), the stimulus duration was set at 150 time units in order to see the elevation in \( V \) caused by the input current.](image-url)
larger values of $I_0$ caused $V$ to rise and then reach a steady value after some damped oscillations. In the large amplitude solutions, the oscillations are found to be periodic after an initial transient. The period of oscillation varies with $I_0$, decreasing to a minimum and then rising again until the periodic solutions disappear. This behavior is shown in Fig. 2 for the parameter values given above. Similar behavior was seen for other parameter values.

![Graph showing the relationship between period and $I_0$](image)

Fig. 2. Pulse period in dimensionless time units plotted against $I_0$ for the periodic solutions to Eqs (2).

2. Electronic Analogue Experiments

We have carried out some experiments using electronic analogues for FitzHugh-Nagumo (Nagumo et al., 1962). The nonlinear device used is a tunnel diode, whose current-voltage characteristic is qualitatively similar to that of nerve membrane and can be approximated by a cubic function (Luzader, 1979; Nagumo et al., 1962). The variable $V$ corresponds to the diode voltage, $W$ corresponds to the inductor current, and parameters $a$ and $b$ are proportional to $\frac{1}{L}$ and $R_L$, respectively (Luzader, 1979). The threshold $\theta$ is controlled by the bias voltage $V_B$ (increasing $V_B$ decreases $\theta$).

The circuit diagram for the "membrane patch" model is shown in Fig. 3, and some characteristic responses are shown in Fig. 4. As the input current is increased from zero, single pulses appear, then oscillation occurs. The pulse frequency increases with increasing current until a maximum frequency is reached, and a further increase in current causes a decrease in frequency before oscillations cease. Frequency vs "stimulus" strength curves are shown in Fig. 5 for two bias settings. It is found that

![Circuit diagram for the electronic analogue](image)

Fig. 3. An electronic analogue of a "membrane patch" based on the FitzHugh-Nagumo model. Current stimulation is achieved by connecting a signal source to the circuit through a large resistance $R_{in}$. The tunnel diode is a 1N3714. A positive bias voltage between 0.11 V and 0.13 V is applied to the circuit. The other component values are $C = 1 \mu F$, $L = 1 \text{ mH}$, $R_L = 30 \Omega$. 
Fig. 4. Diode voltage (bottom trace) of the tunnel diode model to different input currents. The top trace in (a) shows the input voltage pulse. The pulse generator voltage was set at 5 V; $i_0$ was controlled by varying $R_{in}$. The current levels are (a) 0.25 mA, (b) 1.00 mA, (c) 6.25 mA, (d) 6.33 mA, (e) 6.94 mA, (f) 7.14 mA. The bias voltage was set at 0.12 V. The scales in the scope photos are: input pulse, 5 V/div; "nerve" pulses, 0.2 V/div; sweep, 50 μsec/div.

Fig. 5. Pulse frequency of the tunnel diode model vs input voltage. The value of $R_{in}$ was set at 5 kΩ for these measurements. Curves are shown for $V_b = 0.110$ V (A) and 0.120 V (B).
the maximum frequency depends on $L$ (6) and $R_L$ (b) but not on bias (6). The circuit also shows an extra regime of oscillation for values of the applied current above the level at which the oscillations first cease. Eqs (2) show no such tendency, at least for the parameter values investigated so far. Troy's study (1976) also fails to suggest any other values of $I_0$ for which oscillatory solutions would be found for Eqs. (2). This point warrants further study to discover exactly what features in the dynamics of the tunnel diode are responsible for the observed behavior.

To study propagating periodic pulse trains, a series of stages like the one in Fig. 3 can be strung together to form a neuristor as shown in Fig. 6. Then, by injecting a current into the first stage, pulse trains can be observed as they propagate along the circuit. For the experiments discussed here, 10–12 stages were used.

![Circuit Diagram](image)

**Fig. 6.** The circuit diagram for the tunnel diode neuristor. Current stimulation is obtained by connecting the input stage to a pulse generator or power supply through a large resistance. The component values used for the circuit are the same as in Fig. 3 with $r = 1000 \Omega$.

Rinzel and Keller (1973) found that the speed of a propagating periodic pulse train decreased as the pulse frequency increased, and for a given set of model parameters there was a maximum frequency above which no simple periodic travelling wave solutions were found for Eqs (1). By "stimulating" the input end of the neuristor with short pulses of fixed amplitude at different frequencies and measuring the stage-to-stage propagation delay along the circuit, we find that pulse speed does indeed decrease with increasing frequency. Fig. 7 shows propagation speed measured in units of stages/μsec as a function of frequency for three bias voltages. As mentioned above,

![Graph](image)

**Fig. 7.** Pulse speed measured in number of stages/µsec plotted against input pulse frequency for three different bias voltages. $V_B = 0.110 \text{ V (A)}, 0.115 \text{ V (B)}, 0.120 \text{ V (C)}$. For these experiments, 12 stages were used, and the propagation delay between stages 5 and 10 was measured.

changing the bias has the effect of changing threshold for the neuristor: threshold decreases as bias increases. We see that propagation speed increases and the maximum frequency increases as threshold decreases, results that agree with the predictions of Rinzel and Keller (1973).

In addition, the neuristor experiments show what happens when the maximum frequency is exceeded (Fig. 8): the model responds by dividing the input frequency. For example, if the maximum frequency (for the circuit) is 20 kHz, then an input frequency of 28 kHz results in a propagating 14 kHz pulse train; a 45 kHz input gives a 15 kHz pulse train, and so on. For input frequencies near a transition value, the response of the neuristor sometimes appears erratic as viewed on an oscilloscope, or as a patterned pulse train in which rational noninteger division occurs (e.g., two propagating pulses for every three inputs).
When the input signal is a constant current, the input amplitude becomes the parameter that governs the frequency of the propagating pulses. As in the “space clamp” (single stage) experiments, there is an absolute threshold for single pulses, and a higher amplitude for the appearance of periodic pulse trains. Fig. 9 shows a single pulse response to a small current and periodic pulse trains in response to larger currents. In these examples, the input signal is a square current pulse of long duration relative to a neuristor pulse, so the first one or two “action potentials” represent a transient response, after which the circuit settles down to a stable steady state oscillation. As the input current is increased beyond the threshold for oscillation, pulse frequency increases until a maximum is reached, and then the frequency abruptly decreases. This behaviour is also seen in the numerical studies described by Rinzel (1977). For currents near a transition value, patterned pulse trains (periodic groups of two or three pulses) can be produced.

Fig. 10 shows the response of the first and last stages of the neuristor to the long input pulses. (The circuit is essentially unloaded at its output end, being terminated by the high impedance oscilloscope input.) The abrupt decrease in pulse train frequency observed at the output end of the neuristor is due to the failure to propagate of some pulses at the input stage. We also note a reappearance of pulses at large input currents. As mentioned above, this seems to be due to the dynamical properties of the tunnel diode and is being studied further.

3. Discussion

These experiments suggest that the FitzHugh-Nagumo model has an inherent “coding” ability. The results show that the neuristor can “encode” the amplitude of a signal as propagating pulse train, using both mean pulse rate and temporal patterns as information carriers. The code is not unique, however, since different input amplitudes can cause similar pulse trains. The model also has a rather limited dynamic range.

Do these model studies have any physiological implications? Some nerve membranes do show repetitive firing in response to stimulation by a steady current (Holden and Ramadan, 1981). The simulations described here therefore suggest that a coding mechanism is inherent in some excitable membranes owing to their non-linear electrical characteristics. For instance, a cell body would experience a slowly varying depolarization due to the superposition of currents leaking from synaptic inputs in the dendritic tree. This depolarization in turn would cause the somatic membrane to generate action potentials at rates determined by the amplitude of the membrane potential changes, and these pulses would stimulate propagating action potentials on the axon. The net result is a continuously propagating pulse train carrying information about the amplitude of somatic depolarization.
Fig. 9. Response of the neuristor to a sustained input current I_0. The top trace in (a) and (c) is the input voltage pulse; the other traces are the diode voltage measured in the 10th stage. The input voltage was set at 3 V and the current was controlled by varying $R_m$. The currents are (a) 0.25 mA, (b) 0.50 mA, (c) 0.83 mA, (d) 1.00 mA, (e) 1.67 mA, (f) 5.56 mA, (g) 8.33 mA. The scales are the same as in Fig. 4.
Fig. 10. Responses of the first (input) stage of the neuristor and the last (output) stage to the inputs given in Fig. 5. In each photo, the top trace is the response of stage 1, and the bottom trace is the response of stage 10. The input current was controlled as described in Fig. 9, and are (a) 0.83 mA; (b) 1.00 mA; (c) 1.67 mA; (d) 3.56 mA; (e) 8.33 mA; (f) 10.0 mA. The vertical scale is 0.2 V/div for all traces; the sweep is 50 μsec/div.
References

———, preprint.
———, (1976) ibid. 27, 123.