ON THE EXISTENCE OF ALMOST PERIODIC, PERIODIC AND QUASI-PERIODIC SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

TRAN TAT DAT
VIETNAM NATIONAL UNIVERSITY, HANOI (VNU)

Abstract. This note is concerned with the almost periodicity, periodicity, and quasi periodicity of bounded solutions and a so-called Massera criterion for the existence of periodic solution of neutral differential equation with piecewise constant argument.

1. Introduction. The differential equations with piecewise constant arguments (EPCAs, for short) are subjects which are only studied recently. These equations have the structure of continuous dynamical systems in intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals implies a recursion relation for the values of the solution at such points. Therefore, they combine the properties of differential equations and difference equations. These equations are thus similar in structure to those found in certain sequential-continuous models of disease dynamics as treated by Busenberg and Cooke since 1982 (see [3]). The first contribution is due to Cooke and Wiener in 1984 see [5] and Shah and Wiener in 1983 (see [28]). Cooke and Wiener studied in [5, 6] the existence and uniqueness of solution and of its backward continuation on \((-\infty, 0]\) and investigated asymptotic stability of the trivial solution for equation $\dot{x}(t) = Ax(t) + \sum_{k=-N}^{N} A_k x([t+k]) + f(t)$, $x(t) \in \mathbb{R}$. The results about oscillation properties can be found in [30, 31, 32] and references therewith. The existence of periodic solutions has been studied in [7, 29, 30, 32] and references therewith. In 1991, Cooke and Wiener gave a survey paper, (see [7]), of described the results (before 1991) in the area of differential equations with piecewise constant argument, summarizing all the previous work concerning stability, oscillation properties, and existence of periodic solutions. In a series of papers [21, 22, 23, 24], Papaschinopoulos studied the topology equivalence, asymptotic behavior, and integral manifolds for these equations. The study of the existence of almost periodic solutions for EPCA has been studied (see [26, 27, 33, 34, 35]). Recently, quasi periodic solutions and pseudo almost periodic solutions have been studied (see [11, 36, 1]).

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In this note, we consider the neutral differential equation with piecewise constant argument of the form
\[
\dot{x}(t) = Ax(t) + \sum_{k=-N}^{N} A_k x([t + k]) + f(t), \quad x(t) \in \mathbb{R}^m,
\] (1.1)
where \( N \) is a given positive integer, \( A, A_k \) are given real \( m \times m \)-matrices, \( f \) is a bounded function on \( \mathbb{R} \), \([·]\) is the largest integer function.

The main technique of this note is to use the notion of spectrum of a function which has been widely employed in recent researches. We will estimate the spectrum of a bounded function (Carleman, Bohr spectrum), then we will consider the almost periodicity, periodicity, quasi periodicity of solutions. The main results of note are Theorem 3.6, Corollary 3.8, Corollary 3.9, Theorem 3.10 and Theorem 3.12. The estimates of the spectrum of a bounded solution which are obtained in Lemma 3.4, Lemma 3.11 are important. Theorem 3.6, Corollary 3.8 give the spectral conditions for almost periodicity of bounded solutions to Eq. (1.1). Corollary 3.9 gives a spectral condition for periodicity of bounded solutions to Eq. (1.1). Theorem 3.10 shows the existence of periodic solutions of Eq. (1.1). Theorem 3.12 gives a spectral condition for quasi periodicity of bounded solutions to Eq. (1.1).

2. Preliminaries.

2.1. Notation. Thoughout the note, \( \mathbb{Z}, \mathbb{R}, \mathbb{C} \) stand for the sets of integer, real, complex numbers, respectively. \( L^\infty(\mathbb{R}, \mathbb{R}^m), BC(\mathbb{R}, \mathbb{R}^m), BUC(\mathbb{R}, \mathbb{R}^m), AP(\mathbb{R}^m) \) denote the spaces of all \( \mathbb{R}^m \)-valued essential bounded, bounded continuous, bounded uniformly continuous functions on \( \mathbb{R} \) and their subspace including continuous and almost periodic functions, respectively. If \( A \) is a \( m \times m \)-matrix, then the notations \( \sigma(A), \rho(A), \) and \( R(\lambda, A) \) stand for the spectrum, resolvent set, and resolvent of the matrix \( A \). \( sp_C(f), \sigma_b(f) \) denote the Carleman, Bohr spectrum of \( f \), respectively.

2.2. Spectral theory of functions. In this note, we will use the notions of Carleman spectrum and Bohr spectrum of a function.

For an essential bounded function \( u : \mathbb{R} \rightarrow \mathbb{R}^m \), we define Fourier-Carleman transform of \( u \)
\[
\hat{u}(\lambda) = \begin{cases} 
\int_0^\infty e^{-\lambda t} u(t) dt, & Re(\lambda) > 0 \\
-\int_0^\infty e^{\lambda t} u(-t) dt, & Re(\lambda) < 0
\end{cases}.
\]

Obvious, \( \hat{u}(\lambda) \) is holomorphic (analytic) in \( \mathbb{C} \setminus i\mathbb{R} \). Thus, one get the concept of the Carleman spectrum of \( u \) as follows

**Definition 2.1.** We denote \( sp_C(u) = \{ \xi \in \mathbb{R} : \hat{u}(\lambda) \) is not holomorphic in a some neighbourhood of \( \xi \} \), which is said to be Carleman spectrum of \( u \).

**Proposition 2.2.** Let \( \alpha, \beta \in \mathbb{R}, u, v \in L^\infty(\mathbb{R}, \mathbb{R}^m) \). Then

(i) (Linearity) \( (\alpha u + \beta v)(\lambda) = \alpha \hat{u}(\lambda) + \beta \hat{v}(\lambda) \).

(ii) (Transform of derivation) Assume there exists \( \dot{u} \in L^\infty(\mathbb{R}, \mathbb{R}^m) \). Then
\[
\hat{u}(\lambda) = \lambda \hat{u}(\lambda) - u(0).
\]
Fourier coefficients of $f$ for all $\varepsilon > 0$.

**Definition 2.10.** A sequence $f$ for all $\varepsilon > 0$.

**Lemma 2.11.** If $Z \ni \varepsilon$ is called an 

**Definition 2.8.** Give a sequence $\varepsilon$ is called an 

**Definition 2.9.** A function $\varepsilon$ is called an 

**Definition 2.4.** Give a function $\varepsilon$ is called an 

We define the Bohr spectrum of $f$, that is

$$a(f, \lambda) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) dt.$$ 

We define the Bohr spectrum of $f$ as follows

**Definition 2.4.** Give a function $f \in AP(\mathbb{R}^m)$. Let $\lambda \in \mathbb{R}$, then the mean value $a(f, \lambda) := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) dt$ exists and we call $\lambda$ Fourier exponents of $f$, $a(f, \lambda)$ Fourier coefficients of $f$ corresponding to $\lambda$. We denote $\sigma_\lambda(f) = \{\lambda \in \mathbb{R} : a(f, \lambda) \neq 0\}$ and is said to be Bohr spectrum of $f$.

### 2.3. Almost periodic functions and sequences.

In the following, we give some concepts of almost periodic functions and sequences.

**Definition 2.5.** A subset $E \subset \mathbb{R}$ is called a relatively dense set in $\mathbb{R}$ if there exists a number $l > 0$ (inclusion length) such that $[a, a + l] \cap E \neq \emptyset \ \forall a \in \mathbb{R}$.

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**Definition 2.7.** Give a function $f : \mathbb{R} \to \mathbb{R}^m$ and $\varepsilon > 0$. We call the set $T(f, \varepsilon) = \{\tau \in \mathbb{R} : ||f(t + \tau) - f(t)|| < \varepsilon, \ \forall t \in \mathbb{R}\}$ the $\varepsilon$-translation set of $f$ and $\tau$ is called an $\varepsilon$-period for $f$.

**Definition 2.8.** Give a sequence $f : \mathbb{Z} \to \mathbb{R}^m$ and $\varepsilon > 0$. We call the set $TD(f, \varepsilon) = \{\tau \in \mathbb{Z} : ||f(t + \tau) - f(t)|| < \varepsilon, \ \forall t \in \mathbb{Z}\}$ the $\varepsilon$-translation set of $f$ and $\tau$ is called an $\varepsilon$-period for $f$.

**Definition 2.9.** A function $f : \mathbb{R} \to \mathbb{R}^m$ is called an almost periodic function if for all $\varepsilon > 0$ the set $T(f, \varepsilon)$ is a relatively dense set in $\mathbb{R}$.

**Definition 2.10.** A sequence $f : \mathbb{Z} \to \mathbb{R}^m$ is called an almost periodic sequence if for all $\varepsilon > 0$ the $TD(f, \varepsilon)$ is a relatively dense set in $\mathbb{Z}$.

**Lemma 2.11.** If $f(t)$ is a continuous almost periodic function, then $\forall \varepsilon > 0, T(f, \varepsilon) \cap \mathbb{Z}$ is a relatively dense set in $\mathbb{Z}$.
Proof. Using the Bochner criterion for a continuous, almost periodic function \( f(t) \) we have the set \( \{ f_r(\cdot), r \in \mathbb{R} \} \) is relatively compact in \( BC(\mathbb{R}, \mathbb{R}^m) \). It follows that the set \( K = \{ f_r(\cdot), r \in \mathbb{Z} \} \) is relatively compact in \( BC(\mathbb{R}, \mathbb{R}^m) \). Thus, for all \( \varepsilon > 0 \), there exists \( \{ \tau_1, \ldots, \tau_n \} \subset \mathbb{Z} \) such that \( \{ f_{\tau_1}, \ldots, f_{\tau_n} \} \) is an \( \varepsilon \)-net of \( K \).

Putting \( l := \max_{k=1, \ldots, n} \{|\tau_k|\} \), one will prove the inclusion length for \( \varepsilon \) is \( 2l \).

In fact, \( \forall t \in \mathbb{Z}, \exists \tau_k : \| f_t - f_{\tau_k} \| < \varepsilon \). Setting \( \tau := t - \tau_k \) one has

i) \( \tau \in \mathbb{Z} \),

ii) \( \tau \in [t - l, t + l] \),

iii) \( \| f_{\tau} - f \| = \| f_t - f_{\tau_k} \| < \varepsilon \),

i.e. \( \tau \in (T(f, \varepsilon) \cap \mathbb{Z}) \cap [t - l, t + l] \).

So \( T(f, \varepsilon) \cap \mathbb{Z} \) is relatively dense in \( \mathbb{Z} \). \( \square \)

**Corollary 2.12.** If \( f(t) \) is a continuous almost periodic function, then \( \{ f(n), n \in \mathbb{Z} \} \) is an almost periodic sequence.

**Proof.** This has proved by remark that \( T(f, \varepsilon) \cap \mathbb{Z} \subset TD(f, \varepsilon) \). \( \square \)

**Corollary 2.13.** If \( f(t) \) is a continuous almost periodic function, then \( \tilde{f}(t) = f([t]) \) is an almost periodic function.

**Proof.** Corollary has followed by

\[ \forall \tau \in T(f, \varepsilon) \cap \mathbb{Z} \Rightarrow \| \tilde{f}(t + \tau) - \tilde{f}(t) \| = \| f([t] + \tau) - f([t]) \| \leq \varepsilon, \quad \forall t \in \mathbb{R}. \]

\( \square \)

**Lemma 2.14.** If \( f(t) \) is a continuous almost periodic function, then \( \{ h_n \} \) is an almost periodic sequence and \( h(t) \) is an almost periodic function.

where \( h(t) = -\int_t^{t+1} e^{A(t+1-s)} f(s)ds \), \( t \in \mathbb{R} \),

and \( h_n = h(n), \quad n \in \mathbb{Z} \).

**Proof.** For \( \tau \in T(\varepsilon, f) \) one has

\[
\| h(t + \tau) - h(t) \| = \| \int_{t+\tau}^{t+\tau+1} e^{A(t+1-s)} f(s)ds - \int_t^{t+1} e^{A(t+1-s)} f(s)ds \|
\]

\[
= \| \int_0^{1} e^{A(1-s)} [f(t+s+\tau) - f(t+s)]ds \|
\]

\[
\leq \int_0^{1} \| e^{A(1-s)} \| \| f(t+s+\tau) - f(t+s) \| ds
\]

\[
\leq M \varepsilon,
\]

where \( M := \| e^{A} \| \).

Because \( T(\varepsilon, f) \) is relatively dense in \( \mathbb{R} \), \( h(t) \) is an almost periodic function. Thus, from Corollary 2.12 above \( \{ h_n \} \) is an almost periodic sequence and one has the proof. \( \square \)

We have some conditions for the almost periodicity, periodicity of a bounded uniformly continuous function as follows.
Proposition 2.15 (see [2], page 322). Give a function $f \in \text{BUC}(\mathbb{R}, \mathbb{R}^m)$. If $\text{spc}(f)$ is discrete then $f$ is almost periodic.

Lemma 2.16. Assume $f \in \text{BUC}(\mathbb{R}, \mathbb{R}^m)$ and $\text{spc}(f)$ is countable. Then, $f$ is almost periodic.

Lemma 2.17 (see [2], page 323). Let $f \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ and $\tau > 0$. Then, $f$ is $\tau$-periodic if and only if $\text{spc}(f) \subset 2\pi \mathbb{Z}/\tau$.

3. Main results. In this section, we will deal with almost periodicity, periodicity, and quasi periodicity of bounded solutions and the existence of periodic solutions to Eq. (1.1) which is a so-called Massera criterion. First, we take a precise definition of solutions of Eq. (1.1).

Definition 3.1. A function $x : \mathbb{R} \to \mathbb{R}^m$ is said to be a solution of Eq. (1.1) if

(i) $x$ is continuous on $\mathbb{R}$.
(ii) The derivative $\dot{x}(t)$ of $x(t)$ exists everywhere, with the possible exception of the points $n$ ($n \in \mathbb{Z}$), where one-sided derivatives exist.
(iii) $x(t)$ satisfies Eq. (1.1) on each interval $(n, n+1)$, $\forall n \in \mathbb{Z}$.

3.1. The characteristic equation. Assume $x$ is a bounded solution of Eq. (1.1). By using a variation of constants formula Eq. (1.1) is equivalent with

$$x(t) = e^{A(t-n)}x(n) + \sum_{k=-N}^{N} \int_{n}^{t} e^{A(t-s)} ds A_k x(n+k)$$

$$+ \int_{n}^{t} e^{A(t-s)} f(s) ds, \quad (n \leq t < n+1; \forall n \in \mathbb{Z}).$$

(3.1)

Because $x(t)$ is continuous on $\mathbb{R}$, by limiting when $t \to n+1$ one obtains the corresponding difference equation

$$x(n+1) = e^{A}x(n) + \sum_{k=-N}^{N} \int_{0}^{1} e^{A(1-s)} ds A_k x(n+k)$$

$$+ \int_{n}^{n+1} e^{A(n+1-s)} f(s) ds, \quad (\forall n \in \mathbb{Z}).$$

(3.2)

Setting

$$B_0 = e^{A} + \int_{0}^{1} e^{A(1-s)} ds A_0,$$

$$B_1 = \int_{0}^{1} e^{A(1-s)} ds A_1 - I,$$

$$B_k = \int_{0}^{1} e^{A(1-s)} ds A_k, \quad k \in [-1, \pm 2, \ldots, \pm N],$$

$$h_n = - \int_{n}^{n+1} e^{A(n+1-s)} f(s) ds, \quad \forall n \in \mathbb{Z}.$$

Eq. (3.2) become

$$\sum_{k=-N}^{N} B_k x(n+k) = h_n, \quad \forall n \in \mathbb{Z}.$$  

(3.3)
The corresponding homogeneous equation of Eq. (3.3) is
\[
\sum_{k=-N}^{N} B_k x(n + k) = 0, \quad \forall n \in \mathbb{Z}.
\] (3.4)

We will seek solutions of Eq. (3.4) as \(x(n) = e^{n \lambda} c_{\lambda}\), where \(\lambda \in \mathbb{C}\) satisfies
\[
\det \left[ \sum_{k=-N}^{N} B_k e^{(n+k) \lambda} \right] = 0, \quad \forall n \in \mathbb{Z},
\] (3.5)

and \(c_{\lambda}\) is a \(m\)-dimension constant vector such that
\[
\sum_{k=-N}^{N} B_k e^{(n+k) \lambda} c_{\lambda} = 0, \quad \forall n \in \mathbb{Z}.
\] (3.6)

Therefore, the characteristic equation associated with Eq. (1.1) is
\[
\det \left[ \sum_{k=-N}^{N} B_k e^{k \lambda} \right] = 0.
\] (3.7)

In order to comfortable for estimating the spectrum in the following we can give out some notions which relate rigorous to solutions of characteristic equation above.

We call the set of solutions of this characteristic equation the spectrum of Eq. (1.1), denoted by
\[
\Delta = \left\{ \lambda \in \mathbb{C} : \det \left[ \sum_{k=-N}^{N} B_k e^{k \lambda} \right] = 0 \right\}.
\]

Note that \(\lambda \in \Delta\) if and only if \(e^{\lambda}\) is a solution of a polynomial of which order is not larger \(2Nm\). Thus, \(e^{\lambda}\) has not larger \(2Nm\) values and thus \(\Delta\) is a discrete set of form \(\{\lambda_k + 2\pi i \mathbb{Z}, \lambda_k \in \mathbb{C}, k = 1, 2, \ldots, 2Nm\}\).

The image spectrum of equation is a relative interesting character in estimates the spectrum and is defined by
\[
\Delta_i = \{ \xi \in \mathbb{R} : i\xi \in \Delta \}.
\]

Using the spectrum to decomposition of functions is a method which is interested by many people (see [2, 12, 15, 18, 29]). The criterions for periodicity, almost periodic, quasi periodicity, pseudo almost periodicity . . . , of bounded solutions is considered the improve of Massera theorem for general equations and is extented to the theory of functional spaces of admissibility. For EPCA of form (1.1), one obtains also the criterions for almost periodicity, periodicity, quasi periodicity of bounded solutions is due the method of spectral (see Theorem 3.6, Corollary 3.8, Corollary 3.9, Theorem 3.10 and Theorem 3.12).

Remark 3.2. In Fourier-Carleman transform, we only consider \(\{ \lambda : Re\lambda > 0 \}\), otherwise \(\{ \lambda : Re\lambda < 0 \}\) is presented similarly.

An important property of Eq. (1.1) is: each bounded solution is uniformly continuous. This is presented in the following lemma.

Lemma 3.3.

Assume \(x\) is a bounded solution of Eq. (1.1). Then \(x \in \text{BUC}(\mathbb{R}, \mathbb{R}^m)\).
Proof. Put \( M_1 := \max(\|A\|, \|A^{-N}\|, \ldots, \|AN\|) \), 
\( M_2 := \sup_{t \in \mathbb{R}} \|x(t)\| \), 
\( M_3 := \sup_{t \in \mathbb{R}} \|f(t)\| \),
\[ M := M_1 M_2 + (2N + 1) M_1 M_2 + M_3. \]

Uniformly continuity on \( \mathbb{R} \) of a bounded solution \( x(t) \) follows from
\[ \|x(t_2) - x(t_1)\| = \| \int_{t_1}^{t_2} \dot{x}(t) dt \| \]
\[ \leq \int_{t_1}^{t_2} \|\dot{x}(t)\| dt \]
\[ = \int_{t_1}^{t_2} \|Ax(t) + \sum_{k=-N}^{N} A_k x([t + k]) + f(t)\| dt \]
\[ \leq (t_2 - t_1) M, \]
for every \( t_1 < t_2 \) on \( \mathbb{R} \). Lemma has been proved.

Assume \( x(t) \) is a bounded solution of Eq. (1.1), we follow \( x \in BUC(\mathbb{R}, \mathbb{R}^m) \).

Therefore we obtain an expression:
\[ \lambda \tilde{x}(\lambda) - x(0) = A \tilde{x}(\lambda) + \sum_{k=-N}^{N} A_k g_k(\lambda) + \tilde{f}(\lambda), \tag{3.8} \]
where
\[ g_k(\lambda) = \int_{0}^{\infty} e^{-\lambda t} x([t + k]) dt \]
\[ = \frac{1 - e^{-\lambda}}{\lambda} \sum_{n=0}^{\infty} e^{-n\lambda} x(n + k). \tag{3.9} \]

On the other hand, from Eq. (3.2) it follows
\[ x([t + 1]) = e^A x([t]) + \sum_{k=-N}^{N} \int_{0}^{1} e^{A(1-s)} ds A_k x([t + k]) \]
\[ + \int_{[t]}^{[t+1]} e^{A([t+1]-s)} f(s) ds. \tag{3.10} \]

By Fourier-Carleman transform, we have that
\[ g_1(\lambda) = e^A g_0(\lambda) + \sum_{k=-N}^{N} \int_{0}^{1} e^{A(1-s)} ds A_k g_k(\lambda) - \hat{h}(\lambda), \tag{3.11} \]
which is equivalent with
\[ \sum_{k=-N}^{N} B_k g_k(\lambda) = \hat{h}(\lambda), \tag{3.12} \]
where
\[ \hat{h}(t) = h([t]) = - \int_{[t]}^{[t+1]} e^{A([t+1]-s)} f(s) ds. \]
Simple transforms for $g_k(\lambda)$

$$g_k(\lambda) = \frac{1 - e^{-\lambda}}{\lambda} \sum_{n=0}^{\infty} e^{-\lambda}x(n + k)$$

$$= \begin{cases} \frac{1 - e^{-\lambda}}{\lambda} e^{k\lambda} g_0(\lambda) - x(0) - \cdots - e^{-(k-1)\lambda}x(k-1), & k \geq 1 \\ \frac{1 - e^{-\lambda}}{\lambda} e^{k\lambda} g_0(\lambda) + e^{-k\lambda}x(k) + \cdots + e^{\lambda}x(-1), & k \leq -1 \end{cases}$$

$$= \begin{cases} e^{k\lambda} g_0(\lambda) - \frac{1 - e^{-\lambda}}{\lambda} e^{k\lambda}[x(0) - \cdots - e^{-(k-1)\lambda}x(k-1)], & k \geq 1 \\ e^{k\lambda} g_0(\lambda) + \frac{1 - e^{-\lambda}}{\lambda} e^{k\lambda}[e^{-k\lambda}x(k) + \cdots + e^{\lambda}x(-1)], & k \leq -1 \end{cases}$$

$$= e^{k\lambda} g_0(\lambda) + p_k(\lambda), \quad \text{with } p_k(\lambda) \text{ is holomorphic in } \lambda \text{ on } \mathbb{C}.$$
where $r, u$ are vector, matrix-valued functions (respectively) which are holomorphic in $\lambda$ on $\mathbb{C}$ i.e. $\tilde{x}(\lambda)$ is holomorphic in a some neighbourhood of $i\xi$ i.e. $\xi \notin sp_{C}(x)$. Lemma has been proved.

We have some first remarks of the right side of Eq. (3.17):

**Remark 3.5.**

i) $\sigma_{i}(A) = \{\xi \in \mathbb{R} : i\xi \in \sigma(A)\}$ is a finite set on $\mathbb{R}$.

ii) $\Delta_{i} = \{\xi \in \mathbb{R} : i\xi \in \Delta\}$ is a discrete set of form

$$\{\xi_{k} + 2\pi \mathbb{Z}, \xi_{k} \in \mathbb{R}, k \in \{1, \ldots, 2Nm\}\}.$$ 

From lemmas above one gives criterions of almost periodicity of a bounded solution as follows.

**Theorem 3.6.** If a function $f$ satisfies that $sp_{C}(f), sp_{C}(\bar{h})$ are countable then each bounded solution of Eq. (1.1) is an almost periodic solution.

**Proof.** Assume $x$ is a bounded solution of Eq. (1.1). From lemmas 3.3, 3.4 and Remark 3.5 it follows that $x \in BUC(\mathbb{R}, \mathbb{R}^{m})$ and $sp_{C}(x)$ is countable. Lemma 2.16 give that $x(t)$ will an almost periodic solution.

We have a relation between $sp_{C}(f)$ and $sp_{C}(\bar{h})$ as follows

**Lemma 3.7.** If $sp_{C}(f)$ is discrete and the representation $f(t) = \sum \limits_{n \geq 1} a_{n}e^{i\lambda_{n}t}$ for $\sum \limits_{n \geq 1} \|a_{n}\| < \infty, a_{n} \neq 0$ then $sp_{C}(\bar{h})$ is also discrete. Moreover, one has an estimate

$$sp_{C}(\bar{h}) \subset \{\lambda_{n} + 2\pi \mathbb{Z}, n \geq 1\}.$$ 

**Proof.** We have $f(t) = \sum \limits_{n \geq 1} a_{n}e^{i\lambda_{n}t}$ for $\sum \limits_{n \geq 1} \|a_{n}\| < \infty, a_{n} \neq 0$.

Then,

$$\bar{h}(t) = \int_{0}^{1} e^{A(1-s)} f([t] + s)ds$$

$$= \sum \limits_{n \geq 1} \int_{0}^{1} e^{A(1-s)} e^{i\lambda_{n}s} ds a_{n} e^{i\lambda_{n}[t]}$$

$$= \sum \limits_{n \geq 1} b_{n} e^{i\lambda_{n}[t]}.$$
where $b_n = \int_0^1 e^{A(1-s)} e^{i\lambda_n s} ds a_n$. Thus, for $Re \lambda > 0$

$$
\int_0^\infty e^{-\lambda t} \bar{h}(t) dt = \sum_{n \geq 1} b_n \int_0^\infty e^{-\lambda t + i\lambda_n t} dt = \sum_{n \geq 1} b_n \sum_{m \geq 0} \int_{m+1}^{m+1} e^{-\lambda t + i\lambda_n m} dt = \sum_{n \geq 1} b_n \sum_{m \geq 0} e^{(i\lambda_n - \lambda)m} \frac{1 - e^{-\lambda}}{-\lambda} = \sum_{n \geq 1} b_n \frac{1}{1 - e^{(i\lambda_n - \lambda)}} \frac{1 - e^{-\lambda}}{-\lambda}.
$$

For $b_n \neq 0$, it is easy to see $\frac{1}{1 - e^{(i\lambda_n - \lambda)}} \frac{1 - e^{-\lambda}}{-\lambda}$ is holomorphic in a small enough neighbourhood of $i\xi$ for $\xi \notin \{\lambda_n + 2\pi \mathbb{Z}, n \geq 1\}$ (because $\{\lambda_1, \lambda_2, \ldots\}$ is discrete) i.e. $sp_C(\bar{h}) \subset \{\lambda_n + 2\pi \mathbb{Z}, n \geq 1\}$.

Thus, $sp_C(\bar{h})$ is discrete, one has the proof.

Therefore, we have a better estimate for almost periodicity of a bounded solution.

**Corollary 3.8.** If $f$ satisfies that $sp_C(f)$ is discrete and the representation $f(t) = \sum_{n \geq 1} a_n e^{i\lambda_n t}$ for $\sum_{n \geq 1} ||a_n|| < \infty$, $a_n \neq 0$ then a bounded solution of Eq. (1.1) will be an almost periodic solution.

**3.3. Criterion of periodicity of a bounded solution.** From estimate the spectrum above, by the condition "the characteristic equation (3.7) has no solutions on the imaginary axis" and "$\sigma(A) \cap i\mathbb{R} \subset 2\pi \mathbb{Z}$" we have a corollary of periodic solutions.

**Corollary 3.9.** By the condition "the characteristic equation (3.7) has no solutions on the imaginary axis" and "$\sigma(A) \cap i\mathbb{R} \subset 2\pi \mathbb{Z}$" we have

1) If $f$ is periodic of period $p \in \mathbb{N}$ then a bounded solution $x$ is also periodic with the same period (called harmonic solution).

2) If $f$ is periodic of period $p/q$ (where $p, q \in \mathbb{N}$ are relatively prime) then a bounded solution $x$ is periodic of period $p$ (called subharmonic solution).

**Proof.** Function $\bar{h}$ is periodic of period $p$ because

$$
\bar{h}(t + p) = \int_0^1 e^{A(1-s)} f([t + p] + s) ds = \int_0^1 e^{A(1-s)} f([t] + p + s) ds = \int_0^1 e^{A(1-s)} f([t] + s) ds \text{ (as $f$ is periodic of period $p, (p/q)$)} = \bar{h}(t).
$$
From lemma 2.17 it follows that \( sp(f), sp(\bar{h}) \subset \frac{2\pi}{p} \). Then, by the above conditions we have \( \sigma_i(A) \cap \mathbb{R} \subset 2\pi\mathbb{Z}, \Delta_i = \emptyset \) and from Lemma 3.4 we have \( sp_C(x) \subset \frac{2\pi}{p} \) and thus \( x \) is periodic of period \( p \). Corollary has been proved.

3.4. Massera criterion for the existence of a periodic solution. Corollary 3.9 gives us a criterion (is called the Massera criterion) for the existence of periodic solutions as follows

**Theorem 3.10.** Assume Eq. (1.1) satisfies conditions of Corollary 3.9 above. Assumption \( f \) is a periodic function with rational period \( \tau = \frac{p}{q} \). Then Eq. (1.1) has a periodic solution with period \( p \) if and only if it has a bounded solution on the positive half line.

**Proof.** Necessity is obvious. We only have to prove sufficiency. However, from Corollary 3.9 any bounded solution is periodic of period \( p \) so we only have to prove:” if there exists a bounded solution on the positive half line then there exists also a bounded solution on the whole line \( \mathbb{R} \).” In fact, assume that \( x \) is a bounded solution on the positive half line of Eq. (1.1). From the proof of Lemma 3.3 we follow \( x \) is uniformly continuous on the positive half line. Put \( x_n(t) = x(t + np) \), for \( t \geq -np \). Using the Arzel-Ascoli’s theorem one may assume that \( y \in BUC(\mathbb{R}, \mathbb{R}^m) \) is a local uniformly limit of a sequence of functions \( \{x_n\} \) on \( \mathbb{R} \). Give \( t \geq \tau \in \mathbb{R} \). For every \( n \) such that \( np + \tau \geq 0 \) one gets that

\[
x_n(t) - x_n(\tau) = x(np + t) - x(np + \tau) = \int_{np+\tau}^{np+t} (Ax(s) + \sum_{k=-N}^{N} A_k x([s + k]) + f(s))ds
\]

\[
= \int_{np+\tau}^{t} (Ax(np + s) + \sum_{k=-N}^{N} A_k x([np + s + k]) + f(np + s))ds
\]

\[
= \int_{\tau}^{t} (Ax_n(s) + \sum_{k=-N}^{N} A_k x_n([s + k]) + f(s))ds.
\]

Let \( n \to \infty \), the above equation becomes

\[
y(t) - y(\tau) = \int_{\tau}^{t} (Ay(s) + \sum_{k=-N}^{N} A_k y([s + k]) + f(s))ds.
\]

So, \( y \) is a bounded solution on \( \mathbb{R} \) of Eq. (1.1). Therefore theorem has been proved.

3.5. Criterion for quasi periodicity of a bounded solution. Result of Theorem 3.6 above: "a bounded solution is almost periodic" helps us to estimate of Bohr spectrum of a bounded solution.

**Lemma 3.11.** If \( sp_C(f), sp_C(\bar{h}) \) are countable then we have estimate the Bohr spectrum of a bounded solution \( x \) of Eq. (1.1) as follows

\[
\sigma_b(x) \subset \sigma_b(f) \cup \sigma_b(\bar{h}) \cup \sigma_i(A) \cup \Delta_i.
\]
Proof. From Eq. (3.3) we have

$$\sum_{k=-N}^{N} B_k \bar{x}(t + k) = \bar{h}(t). \quad (3.19)$$

By the Theorem 3.6, a bounded solution $x$ is an almost periodic solution. Thus, from corollary 2.13 $\bar{x}$ is also an almost periodic function so Bohr transforms of two sides of Eq. (1.1) exist. Therefore,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} \dot{x}(t) dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} A x(t) dt$$

$$+ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} \sum_{k=-N}^{N} A_k \bar{x}(t + k) dt \quad (3.20)$$

$$+ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) dt.$$

Transform of the left side of Eq. (3.20) by the integrate by parts formula and note that $x$ and $e^{-i\lambda t}$ are bounded we obtain:

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} \dot{x}(t) dt = \lim_{T \to \infty} \frac{1}{2T} e^{-i\lambda t} x(T) \bigg|_{-T}^{T}$$

$$- (i\lambda) \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} x(t) dt$$

$$= i\lambda \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} x(t) dt$$

$$= i\lambda a(x, \lambda).$$

Similarly transform of the right side we obtain

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} A x(t) dt = A \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} x(t) dt$$

$$= A a(x, \lambda)$$

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} \sum_{k=-N}^{N} A_k \bar{x}(t+k) dt = \sum_{k=-N}^{N} A_k \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda(t+k)} \bar{x}(t+k) dt \\
= \sum_{k=-N}^{N} A_k e^{ik\lambda} \lim_{T \to \infty} \frac{1}{2T} \int_{-T+k}^{T+k} e^{-i\lambda t} \bar{x}(t) dt \\
= \sum_{k=-N}^{N} A_k e^{ik\lambda} a(\bar{x}, \lambda) \\
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) dt = a(f, \lambda).
\]

So, Eq. (3.20) will become

\[
(i\lambda - A) a(x, \lambda) = \sum_{k=-N}^{N} A_k e^{ik\lambda} a(\bar{x}, \lambda) + a(f, \lambda). \tag{3.21}
\]

On the other hand, from lemma 2.14 function \( h(t) \) is almost periodic and from Corollary 2.13 \( \bar{h}(t) \) is almost periodic so Bohr transforms of two sides of Eq. (3.19) exist. Thus, by similarly transform we obtain

\[
\sum_{k=-N}^{N} B_k e^{ik\lambda} a(\bar{x}, \lambda) = a(\bar{h}, \lambda). \tag{3.22}
\]

From Eq. (3.22) and the condition \( \lambda \notin \Delta_i \), i.e. \( \sum_{k=-N}^{N} B_k e^{ik\lambda} \) is invertible, we follow

\[ a(\bar{x}, \lambda) = 0 \text{ provided that } a(\bar{h}, \lambda) = 0. \]

For a real number \( \lambda \notin \sigma_b(f) \cup \sigma_b(\bar{h}) \cup \sigma_i(A) \cup \Delta_i \), we have \( a(f, \lambda) = 0, a(\bar{h}, \lambda) = 0 \) and \( (i\lambda - A) \) is invertible thus from Eq. (3.21) it follows that \( a(x, \lambda) = 0 \) or \( \lambda \notin \sigma_b(x) \).

So

\[ \sigma_b(x) \subset \sigma_b(f) \cup \sigma_b(\bar{h}) \cup \sigma_i(A) \cup \Delta_i \]

and lemma has been proved. \( \square \)

From Lemma 3.11 and Remark 3.5 we obtain a criterion for quasi periodicity of a bounded solution.

**Theorem 3.12.** If \( f, \bar{h} \) are quasi periodic functions then each bounded solution of Eq. (1.1) is also quasi periodic.

**Proof.** From the assumption of quasi periodicity of \( f, \bar{h} \), it follows that \( \sigma_i(f), \sigma_b(\bar{h}) \) have bases which are integer and finite. Assume that \( x \) is a bounded solution of Eq. (1.1). By estimating of spectrum in Lemma 3.11 and Remark 3.5 it follows that \( \sigma_b(x) \) has also a basis which is integer and finite. Therefore, the bounded solution \( x \) is quasi periodic. Theorem has been proved. \( \square \)
To illustrate the interesting property of the above Massera criterion for the periodicity of solutions of differential equations with piecewise constant argument we will consider the following examples

**Example 3.13.**

Consider the following equations

(i) \( \dot{x}(t) = -x([t]) + \cos(2\pi t) \cos(2\pi t) \).

(ii) \( \dot{x}(t) = -x([t]) + \cos(4\pi t) \cos(4\pi t) \).

(iii) \( \dot{x}(t) = -x([t]) + \cos(t) \cos(t) \).

In this case the characteristic equation (3.7) now is: \( \det[e^\lambda] = 0 \). It has no solutions, so it has no solutions on the imaginary axis. Simultaneously, \( \sigma(A) \cap i\mathbb{R} = \{0\} \subset 2\pi\mathbb{Z} \).

Therefore the conditions of Corollary 3.9 are satisfied. It follows that

Eq.(i) gives us a unique harmonic solution, that is

\[ x(t) = \frac{1}{2\pi} \left[ \sin(2\pi t) \right]. \]

Eq.(ii) gives us a unique subharmonic solution, that is

\[ x(t) = \frac{1}{4\pi} \left[ \sin(4\pi t) \right]. \]

However, Eq.(iii) has a unique bounded solution (quasi periodic solution) which isn’t periodic

\[ x(t) = ([t] + 1 - t) \left[ \sin([t]) - \sin([t] - 1) \right] + \left[ \sin(t) - \sin([t]) \right]. \]

In fact, by analytic caculation we have \( sp_C(x) = \{1 + 2\pi\mathbb{Z}\} \cup \{-1 + 2\pi\mathbb{Z}\} \). Assume that \( x \) is periodic with a some period \( \tau > 0 \). Then the Lemma 2.17 follows \( sp_C(x) \subset \frac{2\pi\mathbb{Z}}{\tau} \). Thus we have \( 1 = \frac{2\pi n}{\tau} \) and \( 1 + 2\pi = \frac{2\pi k}{\tau} \) where \( n, k \in \mathbb{N} \). This is a contradiction by now

\[ 2\pi = \frac{2\pi k}{\tau} - 1 = \frac{2\pi k}{2\pi n} - 1 = \frac{k}{n} - 1. \]

However, we have \( \sigma_b(x) = sp_C(x) = \{1 + 2\pi\mathbb{Z}\} \cup \{-1 + 2\pi\mathbb{Z}\} \), it follows that \( \sigma_b(x) = \{1 + 2\pi\mathbb{Z}\} \cup \{-1 + 2\pi\mathbb{Z}\} \) i.e. the set of Fourier exponents of \( x \) has a basis \( \{1, 1 + 2\pi\} \) which is integer and finite so \( x \) is a quasi periodic solution.

Therefore, the Massera theorem is true for \( f \) which is periodic of rational period but is not true for \( f \) which is periodic of irrational period.

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Department of Mathematics, Hanoi University of Science, 334 Nguyen Trai, Hanoi, Vietnam.

E-mail address: tatdat4382@yahoo.com