Complex Systems Methods — 9. Critical Phenomena: The Renormalization Group

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Overview

1. The general scheme
   - The renormalization group transformation
   - Flows in parameter space
   - Universality and Crossover
   - Scaling and critical exponents
2. Implementation of the renormalization group
   - The one-dimensional Ising model
   - Higher dimensions — the 2D-Ising model

Scaling, Renormalization and Universality

- Start with some model $M(\boldsymbol{x}, \boldsymbol{p})$ defined on some scale ϵ with parameters $\boldsymbol{p}$.
- Now define new observable $\boldsymbol{x}'$ by coarse graining, e.g. integrating the old ones over a certain range. Then rescale the new variables, such that the model for the new variables is in the same space as the original one, but usually with different parameters $\boldsymbol{p}'$.
- Thus we get a map (or flow) $\boldsymbol{p} \mapsto \boldsymbol{p}'$ in the parameter space, with a semigroup property, the *renormalization group* (RG).
- Self-similar system state $\Rightarrow$ fixed point of the transformation $\Rightarrow$ critical states are unstable fixed points of the RG transformation.
- Stable manifolds of these fixed points represent different models showing the same critical behavior $\Rightarrow$ *universality*
- Critical exponents can be derived from the fixed point properties $\Rightarrow$ they are equal in one *universality class*

What can be explained by the renormalization group?

- Continuous phase transitions fall into universality classes characterized by a given value of the critical exponents.
- For a given universality class there is an upper critical dimension above which the exponents take on mean-field values.
- Relations between exponents, which follows as inequalities from thermodynamics, hold as equalities.
- Critical exponents take the same value as the transition temperature is approached from above or below.

The renormalization group transformation

Literature: J. M. Yeomans, Statistical Mechanics of Phase Transitions, Oxford University Press, 1992.

- Starting point: reduced Hamiltonian $\bar{\mathcal{H}} \equiv \mathcal{H}/kT$
- Renormalization group operator $\boldsymbol{R}$ transforms the reduced Hamiltonian in a new one

$$\bar{\mathcal{H}}' = \boldsymbol{R}\bar{\mathcal{H}}$$

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- The renormalization group operator decreases the number of degrees of freedom from N to N' — either in real space by removing or grouping spins, or in reciprocal space, by integrating out large wavevectors, i.e. removing small wavelength. The *scale factor* of the transformation, b, is defined by

$$b^d = N/N'$$

with d denoting the dimensionality of the system.

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- Scale factor

$$b^d = N/N'$$

- The essential condition to be satisfied by any renormalization group transformation is that the partition function must not change:

$$\mathcal{Z}_{N'}(\bar{\mathcal{H}}') = \mathcal{Z}_N(\bar{\mathcal{H}}) \; .$$

The renormalization group transformation

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- Partition function must not change

$$\mathcal{Z}_{N'}(\bar{\mathcal{H}}') = \mathcal{Z}_N(\bar{\mathcal{H}}) \ .$$

- Reduced free energy per spin (unit volume) $\bar{f} = f/kT$ transforms as

$$\bar{f}(\bar{\mathcal{H}}') = b^d \bar{f}(\bar{\mathcal{H}})$$

Flows in parameter space

- General reduced Hamiltonian

$$\bar{\mathcal{H}} = \sum_{\alpha} \boldsymbol{\mu}.\boldsymbol{\int}$$

with functions (usually products) of *system variables* $\boldsymbol{\int}$ and *conjugated fields* $\boldsymbol{\mu}$.

- E.g. in the case of the Ising model

$$\bar{\mathcal{H}} = -\sum_{i} C - h\sum_{i} s_i - K\sum_{\langle ij\rangle} s_i s_j - J\sum_{\langle ijkl\rangle} s_i s_j s_k s_l - \dots$$

we have $\boldsymbol{\int} = (1, s_i, s_i s_j, s_i s_j s_k s_l, \dots)$,$\boldsymbol{\mu} = (C, h, K, J, \dots)$.

- Fields $\boldsymbol{\mu}$ parametrize the reduced Hamiltonian. Application of the renormalization operator moves the system through parameter space

$$\boldsymbol{\mu}' = \boldsymbol{R}\boldsymbol{\mu}\ .$$

Fixed points of the renormalization operator

$$\boldsymbol{\mu}' = \boldsymbol{R}\boldsymbol{\mu} \qquad \boldsymbol{\mu}^* := \boldsymbol{\mu} \quad \text{with} \quad \boldsymbol{\mu}' = \boldsymbol{\mu} \ .$$

Linearization around the fixed point

$$\begin{aligned} \boldsymbol{\mu} &= \boldsymbol{\mu}^* + \delta\boldsymbol{\mu} \\ \boldsymbol{\mu}' &= \boldsymbol{\mu}^* + \delta\boldsymbol{\mu}' \end{aligned}$$

leads to

$$\delta\boldsymbol{\mu}' = \boldsymbol{A}(\boldsymbol{\mu}^*)\delta\boldsymbol{\mu}$$

with $\boldsymbol{A}$ being the linearization of $\boldsymbol{R}$ at $\boldsymbol{\mu}^*$. Being λ_i and $\boldsymbol{v}_i$ the eigenvalues and eigenvectors of $\boldsymbol{A}$, respectively, we get for two successive transformations with scaling factors b_1 and b_2

$$\lambda_i(b_1)\lambda_i(b_2) = \lambda_i(b_1 b_2)$$

and therefore

$$\lambda_i(b) = b^{y_i} \ .$$

Renormalization near the fixed point

Expand the deviation from the fixed point in terms of the eigenvectors of $\boldsymbol{A}$, $\boldsymbol{v}_i$

$$\boldsymbol{\mu} = \boldsymbol{\mu}^* + \sum_i g_i \boldsymbol{v}_i \ .$$

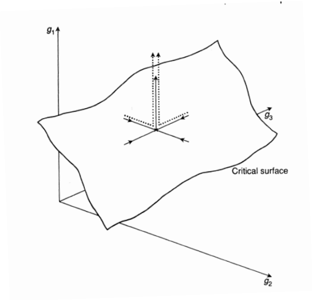
The coefficients g_i are termed the *linear scaling fields*. Applying $\boldsymbol{R}$ leads to

$$\boldsymbol{\mu}' = \boldsymbol{\mu}^* + \sum_i b^{y_i} g_i \boldsymbol{v}_i \quad \text{or}$$

$$g_i' = b^{y_i} g_i \quad \text{respectively.}$$

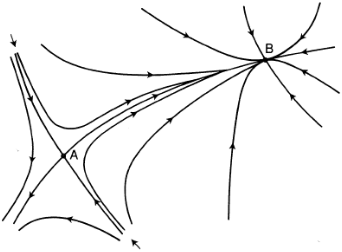
- $y_i > 0$: unstable directions, relevant scaling fields ⇒ control parameters
- $y_i = 0$: marginal stable directions
- $y_i < 0$: stable directions, irrelevant scaling fields ⇒ critical surface, *universality*

Universality



Universality: Under renormalization (scale change) the irrelevant scaling fields will decrease and the system will flow toward the fixed point, while the relevant will increase, driving it away from the critical surface. As long as the relevant fields are initially small enough the trajectory will come close to the fixed point. Therefore its critical behavior will be determined by the linearized transformation at the fixed point and will be independent of the original values of the irrelevant scaling fields.

Crossover



Crossover: If there is more than one fixed point embedded in the critical surface crossover effects may occur. For example in a magnetic system with weak spin anisotropy as the temperature approaches T_c, the system exhibits Heisenberg critical behavior (A), but very close to T_c the critical exponents change to those corresponding to an Ising system (B).

Scaling and critical exponents

The singular part of the rescaled free energy per spin $\bar{f} = f/KT$ was transformed as

$$\bar{f}(\boldsymbol{\mu}) = b^{-d}\bar{f}(\boldsymbol{\mu}')$$

Near the fixed point we have

$$\bar{f}(g_1, g_2, g_3, \ldots) \propto b^{-d}\bar{f}(b^{y_1}g_1, b^{y_2}g_2, b^{y_2}g_2, \ldots)$$

thus $\bar{f}$ is a generalized homogeneous function.
If there are two relevant scaling field (as in the example of the Ising model) we set $g_1 = t = (T - T_c)/T$ and $g_2 = h = H/kT$. Thus

$$\bar{f}(t, h, g_3, \ldots) \propto b^{-d}\bar{f}(b^{y_1}t, b^{y_2}h, b^{y_3}g_3, \ldots)$$

as $t, h, g_3 \to 0$.

Scaling and critical exponents

Free energy:

$$\bar{f}(t, h, 0, \ldots) \propto b^{-d} \bar{f}(b^{y_1} t, b^{y_2} h, 0, \ldots)$$

Specific heat:

$$C \propto \left(\frac{\partial^2 \bar{f}}{\partial t^2} \right)_{h=0} \equiv \bar{f}_{tt}(h = 0) \propto |t|^{-\alpha}$$

leads to

$$\bar{f}_{tt}(h = 0) \propto b^{-d+2y_1} \bar{f}_{tt}(b_1^{y} t, 0) \, .$$

Choosing $b^{y_1}|t| = 1$ gives then

$$\bar{f}_{tt}(h = 0) \propto |t|^{(d-2y_1)/y_1} \bar{f}_t t(\pm 1, 0)$$

and therefore

$$\alpha = 2 - d/y_1$$

Scaling and critical exponents

- Specific heat: $\alpha = 2 - d/y_1$
- Magnetization as function of the temperature: $\beta = (d - y_2)/y_1$
- Susceptibility: $\gamma = (2 - y_2 - d)/y_1$
- Magnetization as function of the magnetic field: $\delta = y_2/(d - y_2)$
- Equations:

$$\alpha + 2\beta + \gamma = 2 \quad \text{corresponds to Rushbrooke inequality}$$
$$\gamma = \beta(\delta - 1) \quad \text{corresponds to Widom inequality}$$

- 2-d Ising Model: $\alpha = 0, \beta = 1/8, \gamma = 7/4, \delta = 15.$

Renormalization for the 1-dimensional Ising model

Reduced Hamiltonian

$$\bar{\mathcal{H}} = -K \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i - \sum_i C$$

Renormalization with $b = 2$:

$$Z = \sum_{\{s\}} \prod_{i=2,4,6,\ldots} \exp\left\{ K s_i (s_{i-1} + s_{i+1}) + h s_i + h(s_{i-1} + s_{i+1})/2 + 2C \right\}$$

Doing the partial trace gives

$$\begin{aligned} Z &= \sum_{s1,s3,\ldots} \prod_{i=2,4,6,\ldots} \{ \exp\left[K(s_{i-1} + s_{i+1}) + h + h(s_{i-1} + s_{i+1})/2 + 2C \right] \\ &\quad + \exp\left[-K(s_{i-1} + s_{i+1}) - h + h(s_{i-1} + s_{i+1})/2 + 2C \right] \} \end{aligned}$$

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Relabeling the spins:

$$\begin{aligned} Z &= \sum_{\{s\}} \prod_i \Big\{ \exp \left[(K + \frac{h}{2})(s_i + s_{i+1}) + h + 2C \right] \\ &\quad + \exp \left[-(K - \frac{h}{2})(s_i + s_{i+1}) + h + 2C \right] \Big\} \end{aligned}$$

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Reduced Hamiltonian

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Renormalization with $b = 2$:

$$\begin{aligned} \exp(K' s_i s_{i+1} + h' s_i + C') \;&=\; \exp\left[(K + \frac{h}{2})(s_i + s_{i+1}) + h + 2C\right] \\ &\quad + \exp\left[-(K - \frac{h}{2})(s_i + s_{i+1}) + h + 2C\right] \end{aligned}$$

leads to

$$\begin{aligned} s_i = s_{i+1} = 1 \quad &: \quad e^{K'+h'+C'} = e^{2K+2h+2C} + e^{-2K+2C} \\ s_i = s_{i+1} = 1 \quad &: \quad e^{K'-h'+C'} = e^{2K-2h+2C} + e^{-2K+2C} \\ s_i = -s_{i+1} = \pm 1 \quad &: \quad e^{-K'+C'} = e^{h+2C} + e^{-h+2C} \end{aligned}$$

Renormalization equations

$$
\begin{aligned}
s_i = s_{i+1} = 1 \quad &: \quad e^{K'+h'+C'} = e^{2K+2h+2C} + e^{-2K+2C} \\
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leads to

$$
\begin{aligned}
e^{2h'} &= (e^{2h} + e^{-4K})(e^{-2h} + e^{-4K})^{-1} \\
e^{4C'} &= e^{8C} e^{4K} (e^{2h} + e^{-4K})(e^{-2h} + e^{-4K}) e^{2h} (1 + e^{-2h})^2 \\
e^{4K'} &= e^{4K} (e^{2h} + e^{-4K})(e^{-2h} + e^{-4K}) e^{-2h} (1 + e^{-2h})^{-2}
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\end{aligned}
$$

and with $x = e^{-4K}$,$y = e^{-2h}$ and $\omega = e^{-4C}$

$$
\begin{aligned}
\omega' &= \frac{\omega^2 x y^2}{(1 + xy)(x + y)(1 + y)^2} \\
x' &= \frac{x(1 + y)^2}{(1 + xy)(x + y)} \\
y' &= \frac{y(x + y)}{1 + yx} \, .
\end{aligned}
$$

$$
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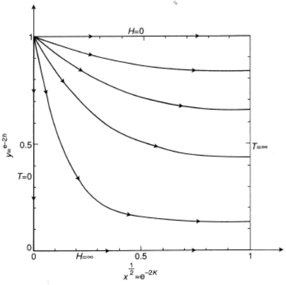
x' &= \frac{x(1+y)^2}{(1+xy)(x+y)} \\
y' &= \frac{y(x+y)}{1+yx}
\end{aligned}
$$

Fixed points for ferromagnetic coupling $K > 0$.

High temperature: $x = 1 \quad 0 \leq y \leq 1$ — infinite temperature, paramagnetic fixed point, attracting

Low temperature and infinite field: $x = 0$ and $y = 0$ — fully aligned configuration

Ferromagnetic fixed point: $x = 0$ and $y = 1$



Scaling at the ferromagnetic fixed point

Linearizing the low equations around the fixed point $(x, y) = (0, 1)$ gives

$$\delta x' = 4\delta x \qquad \delta y' = 2\delta y \ .$$

Hence the eigenvalues of the linearized transformation are

$$\lambda_1 = 4 \qquad \lambda_2 = 2$$

and because of the scale factor $b = 2$ we have

$$y_1 = \frac{\ln \lambda_1}{\ln b} = 2 \qquad y_2 = 1 \ .$$

Problem: $T_c = 0$, thus $t = (T - T_c)/T_c = \infty$ and the usual critical exponents are not defined.

Higher dimensions — the 2D-Ising model

If s denoting the remaining and t the spins that are integrated out one has to consider terms such as

$$\exp K s_{00}(t_{01} + t_{0-1} + t_{10} + t_{-10})$$

Taking the trace over s_{00} gives

$$2\cosh K(t_{01} + t_{0-1} + t_{10} + t_{-10})$$

which can be rewritten as

$$\exp\{a(K) + b(K)(t_{-10}t_{01} + t_{10}t_{01} + t_{10}t_{0-1} \\ + t_{-10}t_{0-1} + t_{-10}t_{10} + t_{0-1}t_{01}) + c(K)t_{-10}t_{01}t_{10}t_{0-1}\}$$

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$$\begin{aligned} a(K) &= \ln 2 + (\ln\cosh 4K + 4\ln\cosh 2K)/8 \\ b(K) &= (\ln\cosh 4K)/8 \\ c(K) &= (\ln\cosh 4K - 4\ln\cosh 2K)/8 \end{aligned}$$

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Taking the trace over s_{00} gives

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$$\bar{\mathcal{H}}' = 2b(K)\sum_{\langle ij\rangle} t_i t_j + b(K)\sum_{[ij]} t_i t_j + c(K)\sum_{sq} t_i t_j t_k t_l$$

with $[ij]$ denoting second neighbors and sq neighbors around a elementary square on the renormlized lattice.

Higher Dimensions

- Starting from next nearest interaction the renormalization procedure generates new (longer range) interaction terms.
- In the case of the 2D-Ising model already the second step of the real space renormalization cannot be made straight forward.
- in general no exact derivation of the renormalization equations possible.
- Approximations are necessary, e.g. Kadanoffs block spin procedure
- ϵ-expansion with respect to the dimension $d = 4 - \epsilon$ in cases where 4 is the upper critical dimension is carried out in k-space
- Numerical methods: Monte-Carlo renormalization group

Outlook: Self-organized criticality (SOC)

- Up to now we only considered the partition function, i.e. no dynamics.
- Dimension of the unstable manifold of the critical fixed point of the renormalization flow — number of control parameters that have to be adjusted to reach the critical state
- SOC: Dynamical system which involves the control parameters and drives them to the critical point