

The Kitaev model and aspects of semisimple Hopf algebras via the graphical calculus

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Based on discussions and work with
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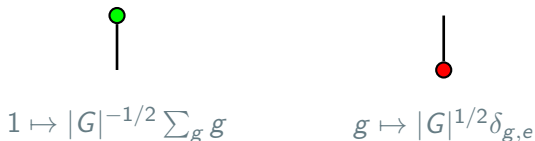
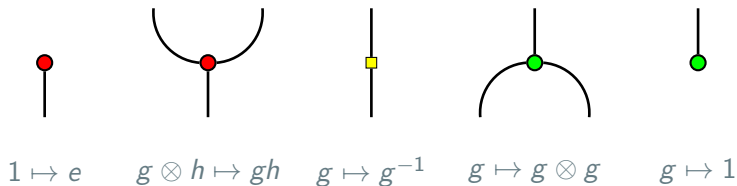
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Overview

Warning: I am new to the field, and this talk is merely a status report without any claims to originality.

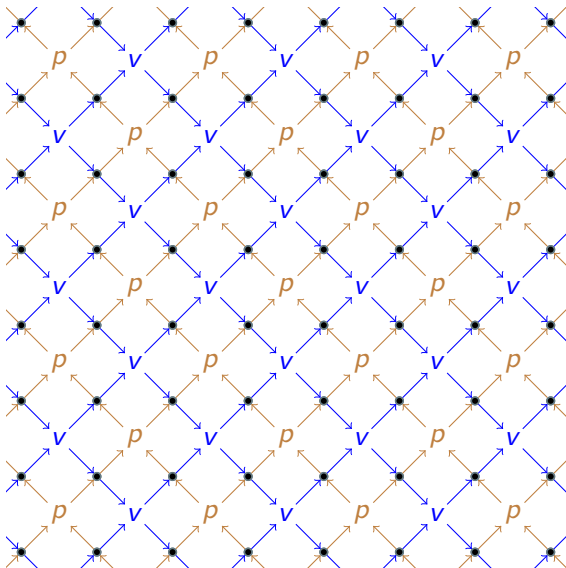
- ▶ The quantum double models from the perspective of the graphical calculus.
- ▶ The graphical calculus of finite-dimensional semisimple Hopf algebras.
- ▶ Some speculation on planar algebras and lifting TQFTs from 2D to 3D.
- ▶ Trying to understand the relationship between the antipode and Haar integral properties for general Hopf monoids in (traced) symmetric monoidal categories.

- ▶ The quantum double models are parametrized by a finite-dimensional semisimple Hopf algebra H , equipped with a (suitably normalized) Haar integral and Haar functional.
- ▶ Thus we have the following pieces of structure:

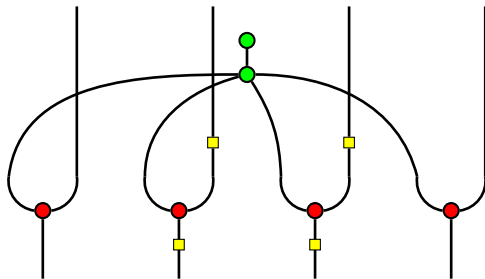


where the intended meaning in the group algebra case illustrates my normalization conventions.

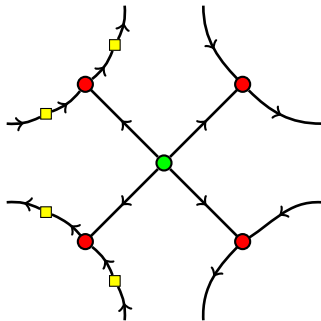
Now let's take a ~~ribbon graph~~ dual pair of graphs underlying the Kitaev model on a surface:



At each vertex, and up to a factor of $\bigcirc^{-1/2}$, we have the operator



We can also draw it in the plane—if we keep track of the orientation of the wires!

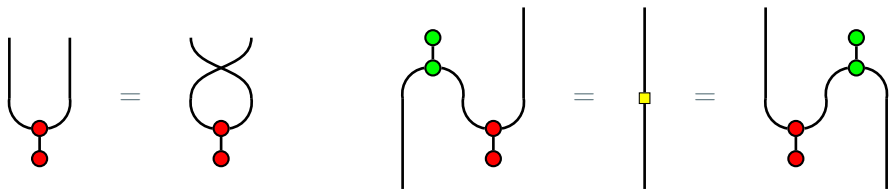


The plaquette operators are the same, but with the colours reversed. Since the vertex and plaquette operators commute, we can compose them in any order by matching up the external terminals. For example like this:

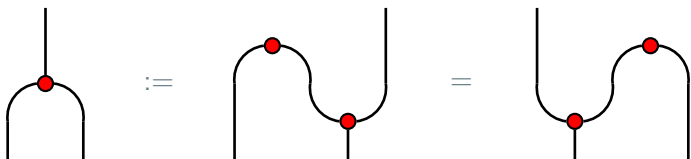
- ▶ One pair of wires at each qudit system remains unmatched: the overall diagram still needs to define an operator!
- ▶ The overall composite operator is the projector onto the ground state space.
- ▶ Evaluating it on a closed surface results in closed cycles.
⇒ We need to be in a compact closed category.
- ▶ We can obtain the ground state space dimension by matching up the remaining pairs as well, so that the diagram evaluates to a scalar.
- ▶ In this picture, we have apparently not used a ribbon graph: instead, the surface is described by a pair of graphs with the same set of edges, playing the roles of Poincaré duals of each other.

What properties of finite-dimensional semisimple Hopf algebras have we actually used for this? In other words, what is their graphical calculus?

It's well-known that a finite-dimensional semisimple Hopf algebra is also a symmetric Frobenius algebra, via the Haar functional evaluated on a product as the bilinear form,

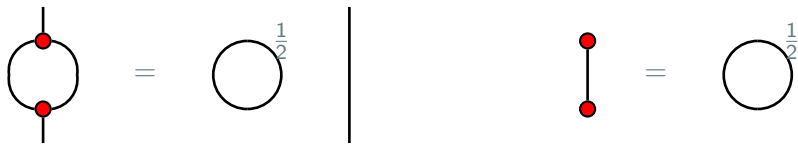


Thus we have a comultiplication,

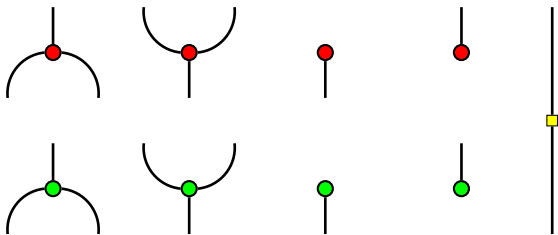


forming a comonoid and satisfying the Frobenius law.

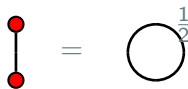
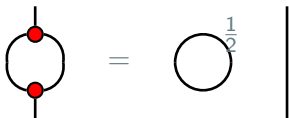
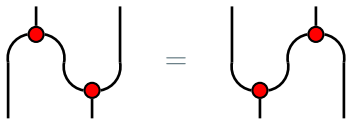
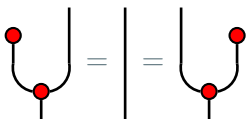
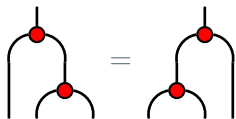
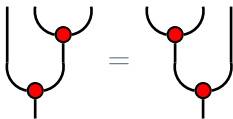
By strong separability, we can normalize to achieve **quasi-specialty** and **quasi-extra-specialty**,



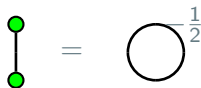
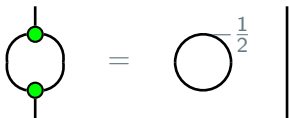
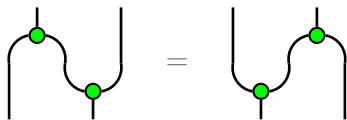
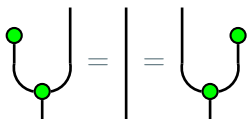
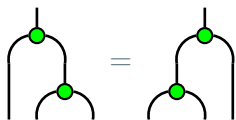
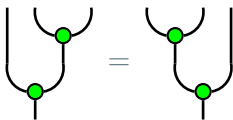
So overall, in a finite-dimensional semisimple Hopf algebra, we have the following pieces of structure:



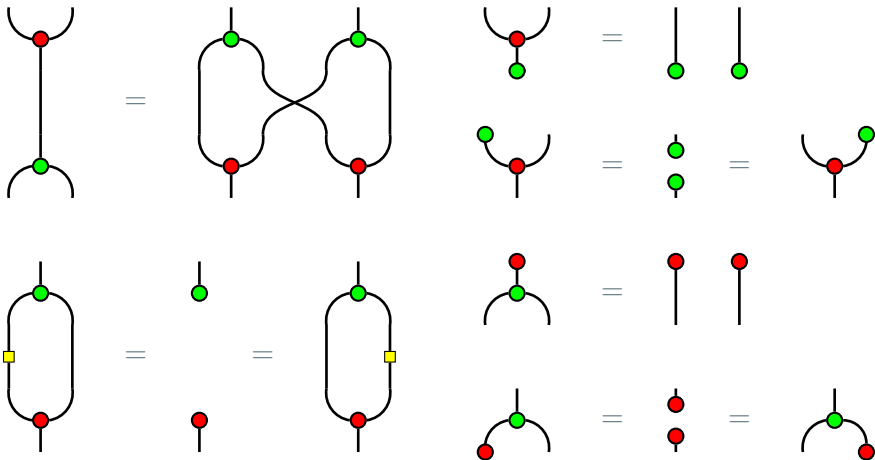
such that each colour forms a quasi-special Frobenius algebra,



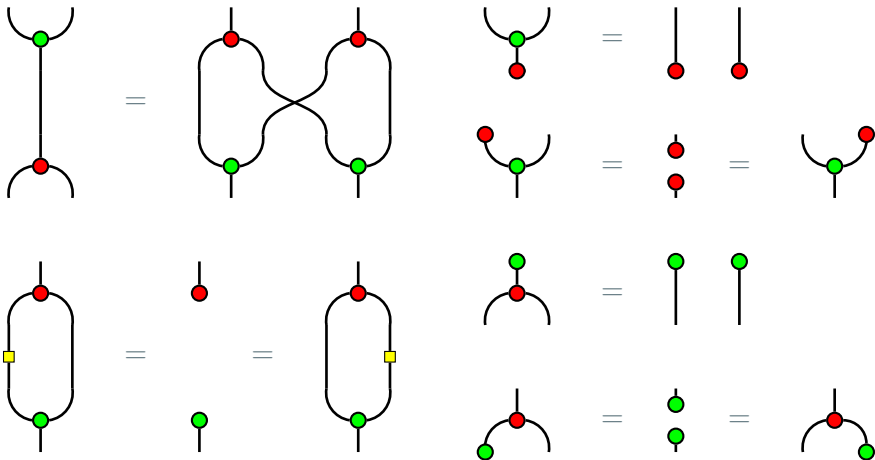
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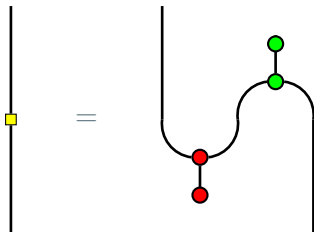
And such that both pairs of colours interact as in a Hopf algebra with Haar integral and functional,



And such that both pairs of colours interact as in a Hopf algebra with Haar integral and functional,



- ▶ It would have been enough to impose this interaction for one pair of colours.
- ▶ Also the antipode is redundant,



so that we equivalently have a pair of symmetric Frobenius algebras, interacting via bialgebra and Haar properties.

- ▶ All of this is intimately related to the **interacting Hopf algebras** of [Bonchi-Sobociński-Zanasi '15](#) and the **interacting Frobenius algebras** of [Duncan-Dunne '16](#).
- ▶ However, our setup is weaker in that we do not assume commutativity or a dagger.
- ▶ We still need to find minimal sets of relations.

In summary, we have:

Proposition (preliminary)

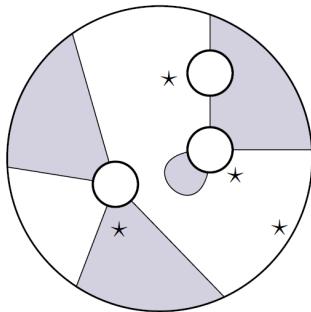
The following pieces of structure uniquely determine each other:

- ▶ A model of the above graphical calculus.
- ▶ A finite-dimensional semisimple Hopf algebra.
- ▶ A pair of quasi-special symmetric Frobenius algebras interacting via bialgebra and Haar properties.
- ▶ A pair of Hopf algebras interacting via Frobenius (and Haar?) properties.

- ▶ Models of this calculus are very rigid: every morphism is an isomorphism!
- ▶ Compare: every morphism of TQFTs is an isomorphism.
- ▶ Every Hopf algebra has a 1-parameter group of trivial deformations given by rescalings. The (extra-)specialty equations that we impose guarantee that the same holds for a model of this calculus.
- ▶ Group of discrete symmetries: $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ corresponding to taking (co-)opposites and swapping colours.

The Wacky Slide

It is known that every semisimple Hopf algebra defines a planar algebra (Kodiyalam-Sunder '05), with syntax governed by planar tangles:



By Emily Peters - [arXiv:0902.1294](https://arxiv.org/abs/0902.1294), CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=57707668>

Question

What is the relationship to our graphical calculus? Do the two shadings correspond to our two colours? And which planar algebras arise from finite-dimensional semisimple Hopf algebras?

The Other Wacky Slide

- ▶ It is known that separable symmetric Frobenius algebras are the same thing as $(2,0)$ -TQFTs (Schommer-Pries '11).
- ▶ Assuming that separable = quasi-special, our calculus therefore is about a pair of interacting $(2,0)$ -TQFTs!
- ▶ And the quantum double models or Turaev-Viro construction turn this into a $(3,0)$ -TQFT! This leads to:

Question

Is there a topological construction turning a pair of suitably compatible $(2,0)$ -TQFTs into a $(3,0)$ -TQFT?

- ▶ Perhaps the earlier picture of a pair of Poincaré dual graphs may help here.

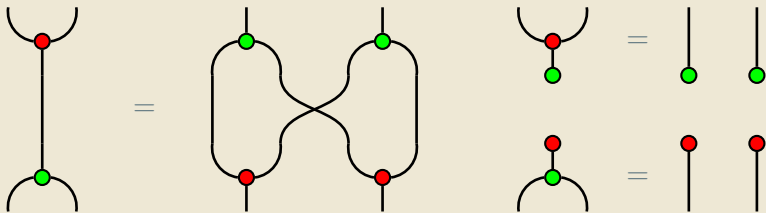
Question

Is it possible to evaluate a $(2,0)$ -TQFT on a *graph*? I.e. can one extend from 1-manifolds to graphs via a colimit construction?

It would be great to understand the models of our graphical calculus in greater generality, which requires understanding the interplay of the antipode and Haar integrals.

Proposition

Let A be a bimonoid in a symmetric monoidal category,



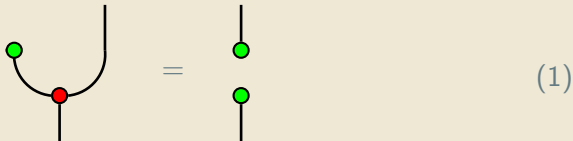
If A has a non-singular left Haar integral, then A has an antipode.

This and the following results are very similar to those of [Kuperberg '97](#)—and probably both strictly weaker and less elegant. But we hope to generalize to the braided case and to weak Hopf algebras.

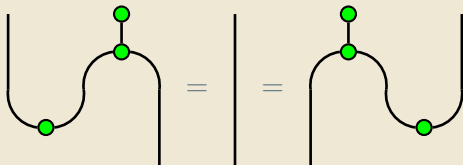
Here, we have used:

Definition

For a bimonoid A , a **left Haar integral** is an element \int^l such that


$$\int^l = \int^l \int^l \quad (1)$$

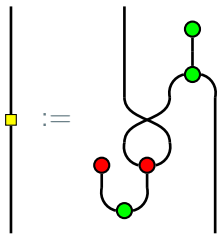
holds. It is **non-singular** if there is \int^r such that the zig-zag equations


$$\int^l \int^r = \int^l = \int^r \int^l$$

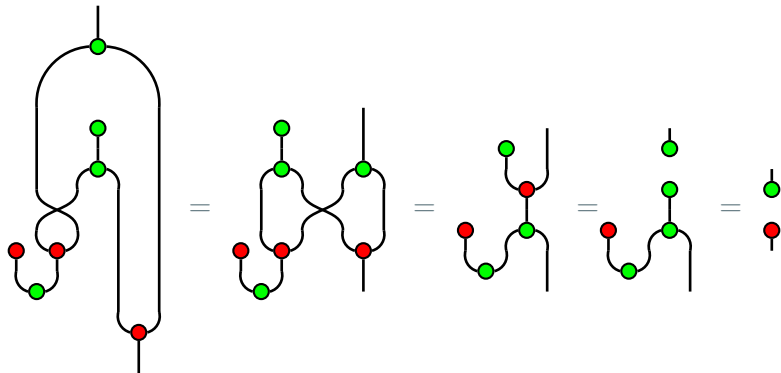
hold.

Note the similarity between (1) and the multiplicativity of the counit!

Let us prove the proposition: Upon defining a candidate antipode as



we can check the left antipode law,

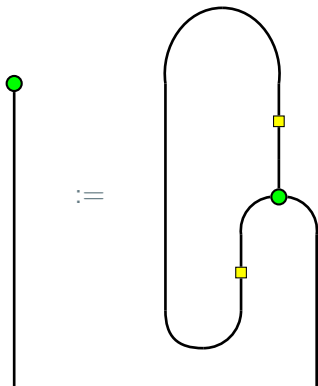


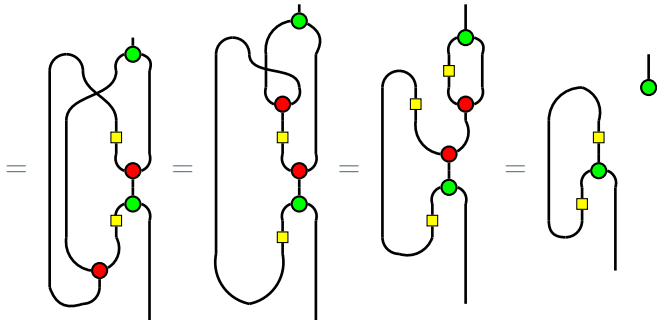
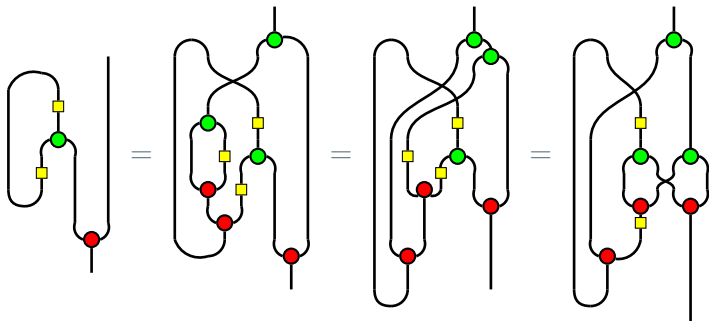
A right antipode can be constructed as the mirror image of the above diagram. We are done since a left and right antipode must coincide.

Proposition

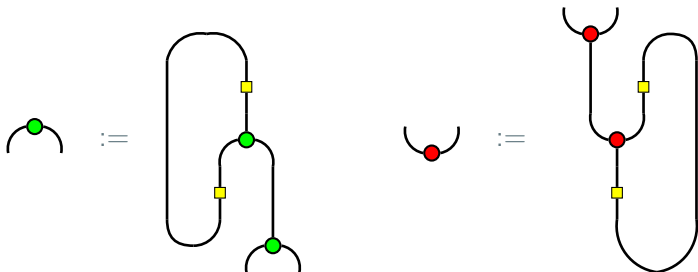
Suppose that we have a Hopf monoid A in a traced monoidal category such that $\text{Tr}(S^2)$ is an invertible scalar. Then there is a non-singular left Haar integral, and S is invertible.

Proof:

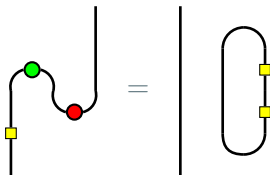




For invertibility of the antipode, we can introduce



and then another calculation shows that



So if $\text{Tr}(S^2)$ is invertible, we have a left inverse of the antipode. It having a right inverse is similar, and this proves both the invertibility of S and the non-singularity of the Haar integral.