Equality

Tobias Fritz

Perimeter Institute for Theoretical Physics

Colloquium, May 2014

Equations

As physicists, we use equality in terms of **equations** on a daily basis. However, depending on the context, the meaning varies:

- Equality of numbers: $2^2 = 4$,
- Equality of functions: $(x + 1)^2 = x^2 + 2x + 1$,
- Equality of groups: $\mathrm{Spin}(6) = SU(4)$,
- Equality of points in spacetime.
- ...

But what is actually the meaning of this? How is equality defined?

Equality as it is traditionally defined

In the conventional foundations of mathematics, every mathematical object is a set. Every number is a set, and so is every function, every group, manifold, point in a manifold etc.

Axiom of extensionality: two sets are equal if and only if they have the same elements,

$$A = B \quad \text{if and only if} \quad \forall x (x \in A \Leftrightarrow x \in B).$$

In particular, this means that there is a notion of equality between the **elements** of a set, since these are themselves sets.

$\Rightarrow$ We are used to assuming that every mathematical object comes automatically equipped with an underlying notion of equality.

My point: assuming this is very often misleading and makes things look more complicated than they are.

Equality as it should be defined

It is better *not* to assume that every set comes equipped with a notion of when two of its elements are equal. Instead of thinking of a set A as a discrete collection of distinct elements $x, y \in A$, it is better to think of it as an amorphous mass without any a priori way of making sense of the question "is $x = y$?"

Nevertheless, we still want to use equations and compute with them! So where could a notion of equality come from? The key idea:

> Equality should be **defined in terms of the other structure** present on a mathematical object.

Example: causal sets

In the causal set approach to quantum gravity, a causal set is a set C equipped with a binary relation $\preceq$ satisfying

1. reflexivity: $x \preceq x$,
2. transitivity: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$,
3. antisymmetry: if $x \preceq y$ and $y \preceq x$, then $x = y$.
4. local finiteness: there are only finitely many y with $x \preceq y \preceq z$.

For us, the antisymmetry axiom is key: with our point of view, it is **not an axiom**, but rather the **definition** of equality:

$$x = y \quad \text{if and only if} \quad x \preceq y \text{ and } y \preceq x.$$

In physics terms: any points in a causal loop are actually the same point.

Also, the local finiteness axiom uses a notion of equality!

Example: preparation and measurement procedures

Let $\mathcal{P}$ be a collection of *preparation* procedures on a physical system and $\mathcal{M}$ a collection of *measurement* procedures on that system.

Combining any preparation $p \in \mathcal{P}$ with any measurement $m \in \mathcal{M}$ results in a probability distribution $P(x|p, m)$ over the outcomes of M.

Two preparations are defined to be equal if they give the same probabilities under any measurement,

$$p_1 = p_2 \quad \text{if and only if} \quad P(x|p_1, m) = P(x|p_2, m) \quad \forall x, m,$$

and similarly for equality of measurements,

$$m_1 = m_2 \quad \text{if and only if} \quad P(x|p, m_1) = P(x|p, m_2) \quad \forall x, p.$$

$\Rightarrow$ Two procedures may be equal even though they are implemented differently in the laboratory. This is analogous to how two functions may be equal, $(x+1)^2 = x^2 + 2x + 1$, despite being given by different algebraic expressions.

Example: social equality

"Social equality is a state of affairs in which all people within a specific society or isolated group have the same status in certain respects."

There are different ways to make this more concrete:

- "Equality of outcome [..] describes a state in which people have approximately the same material wealth or in which the general economic conditions of their lives are similar."
- "Equal opportunity is a stipulation that all people should be treated similarly, unhampered by artificial barriers or prejudices or preferences [..]"

One may also restrict to comparing particular groups of people: gender equality, racial equality, etc.

(All quotes from Wikipedia.)

Example: metric spaces

Let X be a set equipped with a distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

satisfying $d(x, x) = 0$ and the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

We can now define

$$x = y \text{ if and only if } d(x, y) = 0.$$

In other words, the often-imposed axiom

$$d(x, y) = 0 \Leftrightarrow x = y$$

can be reinterpreted as the **definition** of $x = y$.

A general pattern

Leibniz's law: **identity of indiscernibles**,

$$x = y \quad \text{if and only if} \quad \forall P\,(P(x) \Leftrightarrow P(y)).$$

This is a useful notion of equality in many mathematical and non-mathematical contexts. It depends on the collection of properties P that are to be considered!

$\Rightarrow$ Equal things can be substituted for each other in any property P:

$$\text{if } x = y \text{ and } P(y), \text{ then } P(x).$$

Example: consider the property 'being equal to z' for some third thing z. In this case, we get

$$\text{if } x = y \text{ and } y = z, \text{ then } x = z,$$

which is the transitivity of equality.

A general pattern

Typically, applying identity of indiscernibles to a mathematical object means: two elements of the object are equal if and only if they cannot be distinguished by the structure on that object.

- If $x \preceq y \preceq x$ in a causal set, then x and y are physically indistinguishable.
- If $P(x|p_1, m) = P(x|p_2, m)$ for preparation procedures p_1, p_2, then these are operationally indistinguishable.
- If $d(x, y) = 0$ in a metric space, then x and y are geometrically indistinguishable, since $d(x, z) = d(y, z)$ for all z is a consequence.
- If two people have the same wealth, then they are economically indistinguishable. Which properties correspond to 'equal opportunity'?

Example: equality of groups

If we apply identity of indiscernibles to two groups, then we get the definition: two groups G and H are equal if and only if they satisfy the same logical sentences, e.g.

$$\forall x \in G, \exists y \in G, x = y^2 \quad \Longleftrightarrow \quad \forall x \in H, \exists y \in H, x = y^2.$$

However, there exist G and H that are equal in this sense despite not being isomorphic.

But what we mean by an equation like $\mathrm{Spin}(6) = SU(4)$ is the existence of an *isomorphism* $\mathrm{Spin}(6) \cong SU(4)$.

Is there a notion of equality of groups or other objects which comprises isomorphism?

$\Rightarrow$ If so, we should give up the assumption that two objects are either equal or not, and possibly permit them to be equal in more than one way!

Equality in Homotopy Type Theory

Homotopy Type Theory (HoTT) is an emerging branch of mathematics which can serve as a foundation alternative to logic and set theory. One of its features is its elegant treatment of equality.

The following exposition is an informal illustration of equality in HoTT, using more conventional language.

Let **Spaces** be the collection of all topological spaces. In HoTT, **Spaces** is a space in its own right.

For $A \in \textbf{Spaces}$ and $x, y \in A$, write

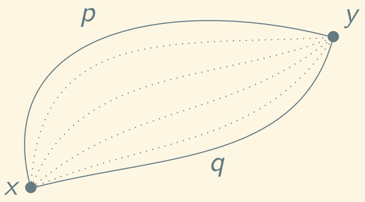
$$x = y \in \textbf{Spaces}$$

for the **space of paths** from x to y in A.

Equality in Homotopy Type Theory

In general, there are many paths from x to y, and hence there are many ways in which x and y can be 'equal'.

For two paths $p, q \in x = y$, one can consider the space of all paths $p = q$. The points of this space are 'paths between paths' or **homotopies** between p and q:



And so on for paths between paths between paths etc.

You may already be familiar with this from algebraic topology.

Equality in Homotopy Type Theory

Previously, we considered properties P defined for the elements of a mathematical object A. Such a property was a function

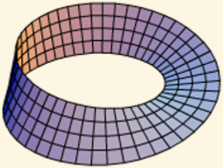
$$P : A \to \{\text{True, False}\}$$

assigning to every $x \in A$ a truth value $P(x)$.

But now for $A \in \textbf{Spaces}$, we can also consider functions

$$P : A \to \textbf{Spaces},$$

which corresponds to a family of spaces $P(x) \in \textbf{Spaces}$, continuously varying in $x \in A$. For example, a Möbius strip is a family of spaces varying over the circle:



Equality in Homotopy Type Theory

The inference rules of HoTT yield a variant of the previous 'identity of indiscernibles' principle, in the guise of a function

$$(x = y) \quad \longrightarrow \prod_{P : A \to \mathbf{Spaces}} (P(x) \to P(y))$$

which, for every family $P : A \to \mathbf{Spaces}$, turns a path $p \in x = y$ into a function $P(x) \to P(y)$.

$\Rightarrow$ Every family of spaces $P : A \to \mathbf{Spaces}$ comes automatically equipped with a notion of **parallel transport**.

One might think that this relates HoTT to gauge theory. It does seem possible to do gauge theory in HoTT, but in a different way that I don't understand (yet).

Isomorphism in Homotopy Type Theory

Previously, we also wrote an isomorphism of groups, $\mathrm{Spin}(6) = SU(4)$, as an equality.

In HoTT, we will see that this is perfectly accurate. For simplicity, consider spaces $A, B \in \mathbf{Spaces}$ instead of groups.

An isomorphism $A \cong B$ consists of maps $f : A \to B$ and $g : B \to A$ such that

$$\forall x \in A,\ g(f(x)) = x, \qquad \forall y \in B,\ f(g(y)) = y.$$

But now these equalities are homotopies—and hence an 'isomorphism' in HoTT is actually a homotopy equivalence!

Just as there is a space of equalities $(A = B) \in \mathbf{Spaces}$, there also is a space of homotopy equivalences $(A \cong B) \in \mathbf{Spaces}$.

Isomorphism in Homotopy Type Theory

Roughly, the **univalence axiom** of HoTT states that the space of equalities and the space of equivalences are themselves equivalent,

$$(A = B) \cong (A \cong B).$$

In particular, every isomorphism can be turned into an actual equality.

One can show that this not only holds for spaces, but also for spaces equipped with additional structure.

In this way, one can make perfect sense of regarding an isomorphism of groups $\mathrm{Spin}(6) \cong SU(4)$ as an equality of groups $\mathrm{Spin}(6) = SU(4)$.

Isomorphism in Homotopy Type Theory

Example application:

if P : **Groups** $\to$ **Spaces** assigns to every group its space of representations, then identity of indiscernibles

$$(\mathrm{Spin}(6) = SU(4)) \longrightarrow \prod_{P:\mathbf{Groups}\to\mathbf{Spaces}} (P(\mathrm{Spin}(6)) \to P(SU(4)))$$

can be applied to construct a function which turns every representation of $\mathrm{Spin}(6)$ into a representation of $SU(4)$.

$\Rightarrow$ We used parallel transport on the space of groups!

Let me end with something less abstract. . .

Food for thought

At the Planck scale, spacetime is generally thought to behave very differently from the macroscopic scale.

Some fundamental questions about spacetime at the Planck scale:

- Does it still make sense to talk about points of spacetime?
- Assuming that it does, then does it still make sense to talk about equality of points? What kind of equality should it be?
- Could the non-traditional nature of equality account for the expected 'blurriness'?

The latter is likely to be a naive idea. Can you tell me why?

Summary

Main lessons:

- Traditionally, every set comes equipped with its own notion of equality of elements.
- This point of view is often not useful.
- It is more appropriate to define equality of elements of a mathematical object in terms of the other structure on this object.
- With some definitions of equality, it happens that two things can be equal in more than one way.
- In conclusion: don't hesitate to redefine equality as it fits your needs.