Quantum correlations and group C^*-algebras

based on

Tsirelson's problem and Kirchberg's conjecture, arXiv:1008.1168

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Overview

Other stuff more interesting than this:

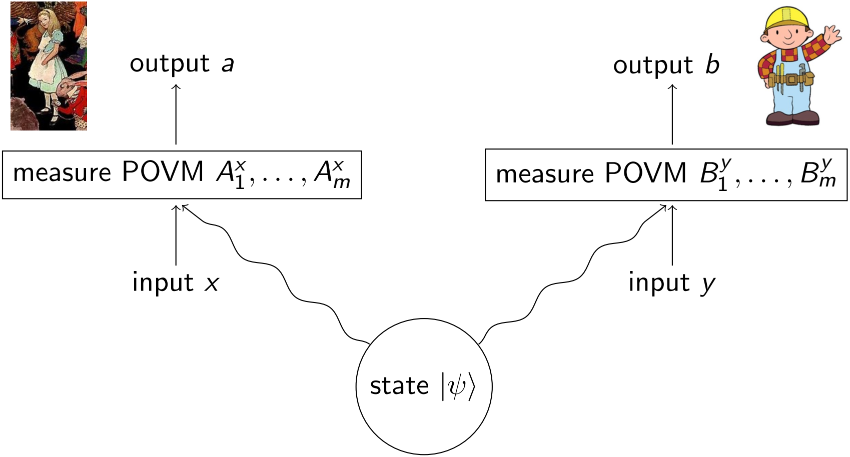
Bell's Theorem and Bayesian Networks, `http://pirsa.org/12020134`

This talk:

1. Bell scenarios and quantum correlations
2. Quantum correlations in terms of group C^*-algebras
3. First applications:
   - All quantum correlations in the CHSH scenario arise from qubits (Masanes '04)
   - Tsirelson's problem
   - Hierarchy of semidefinite programs characterizing quantum correlations (NPA '08)

Bell scenarios

$(2, k, m)$ scenario: 2 parties, for each k POVMs, m outcomes each.



Quantum correlations:

$$P(a, b|x, y) = \langle\psi|(A_a^x \otimes B_b^y)\psi\rangle$$

Quantum correlations and group C^*-algebras I

- By adding ancillas, the measurements can be made projective:

$$A_a^x \cdot A_{a'}^x = \delta_{aa'} A_a^x, \qquad B_b^y \cdot B_{b'}^y = \delta_{bb'} B_b^y.$$

- Label the outcomes with roots of unity

$$e^{2\pi i \cdot \frac{1}{m}}, \ldots, e^{2\pi i \cdot \frac{m}{m}}$$

  so that the measurements are described by unitaries of order m:

$$(U_x)^m = \mathbb{1} = U_x^* U_x = U_x U_x^*$$

- Then specifying the measurements is equivalent to specifying unitaries of order m,

$$U_1, \ldots, U_k; \qquad V_1, \ldots, V_k.$$

Quantum correlations and group C^*-algebras II

- Then specifying the measurements is equivalent to specifying unitaries of order m,

$$U_1, \ldots, U_k; \qquad V_1, \ldots, V_k.$$

- A unitary of order m is the same as a unitary representation of the cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$,

$$\mathbb{Z}_m \to \mathcal{U}(\mathcal{H}), \qquad [r] \mapsto U^r.$$

- k unitaries of order m are the same as a unitary representation of the group

$$\Gamma = \underbrace{\mathbb{Z}_m * \ldots * \mathbb{Z}_m}_{k \text{ factors}}.$$

- For each party, specifying the observables is equivalent to specifying a unitary representation:

$$\pi : \Gamma \longrightarrow \mathcal{U}(\mathcal{H}).$$

Group C^*-algebras I

- For a group Γ, the group algebra $\mathbb{C}[\Gamma]$ is the vector space with basis $\{\delta_g,\ g \in \Gamma\}$ and multiplication defined by

$$\delta_g \delta_{g'} = \delta_{gg'}$$

  and extending bilinearly. A generic element of $\mathbb{C}[\Gamma]$ is $\sum_{g\in\Gamma} c_g \delta_g$ with finitely many coefficients $c_g \neq 0$.
- Group representations $\Gamma \to GL(V)$ correspond to algebra representations $\mathbb{C}[\Gamma] \to \mathrm{End}(V)$.
- On $\mathbb{C}[\Gamma]$, introduce the involution $*$ and the norm $||\cdot||$,

$$\left(\sum_g c_g \delta_g\right)^* = \sum_g \overline{c}_{g^{-1}} \delta_g, \qquad \left\|\sum_g c_g \delta_g\right\| = \sup_{\pi:\Gamma\to\mathcal{U}(\mathcal{H})} \left\|\sum_g c_g \pi(g)\right\|$$

  Define $C^*(\Gamma)$ to be the completion of $\mathbb{C}[\Gamma]$. It is a C^*-algebra.

Group C^*-algebras II

- On the algebra, introduce the involution $*$ and the norm $||\cdot||$,

$$\left(\sum_g c_g \delta_g\right)^* = \sum_g \overline{c}_{g^{-1}} \delta_g, \quad \left|\left|\sum_g c_g \delta_g\right|\right| = \sup_{\pi : G \to \mathcal{U}(\mathcal{H})} \left|\left|\sum_g c_g \pi(g)\right|\right|$$

  Define $C^*(\Gamma)$ to be the completion of $\mathbb{C}[\Gamma]$. It is a C^*-algebra.

- Unitary representations $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ correspond to $*$-representations $\pi : C^*(\Gamma) \to \mathcal{B}(\mathcal{H})$.
- For each party, choosing k observables with m outcomes on $\mathcal{H}$ then corresponds to a $*$-representation

$$C^*(\Gamma) \longrightarrow \mathcal{B}(\mathcal{H}).$$

  The projectors A_a^x are the images of fixed elements $e_a^x \in C^*(\Gamma)$.

Quantum correlations in terms of group C^*-algebras I

- Technically easier assumption: take observables A_a^x and B_b^y to live on the same $\mathcal{H}$ with $[A_a^x, B_b^y] = 0$; **commutativity assumption**. Then quantum correlations are of the form

$$P(a, b|x, y) = \langle\psi|A_a^x B_b^y \psi\rangle.$$

- Choosing such observables for both Alice and Bob corresponds to a $*$-homomorphism

$$\pi : C^*(\Gamma \times \Gamma) \longrightarrow \mathcal{B}(\mathcal{H}).$$

  The projections A_a^x and B_b^y correspond to the images of fixed elements $e_a^x, f_b^y \in C^*(\Gamma \times \Gamma)$.

- A state $|\psi\rangle \in \mathcal{H}$ can be pulled back to a C^*-algebraic state ρ_ψ on $C^*(\Gamma \times \Gamma)$,

$$\rho_\psi(\gamma) = \langle\psi|\pi(\gamma)\psi\rangle.$$

  By construction, $\rho_\psi(e_a^x f_b^y) = \langle\psi|A_a^x B_b^y \psi\rangle$.

Quantum correlations in terms of group C^*-algebras II

Theorem

Correlations $P(a, b|x, y)$ are quantum (with the commutativity assumption) iff there is a C^*-algebraic state ρ on $C^*(\Gamma \times \Gamma)$ such that

$$P(a, b|x, y) = \rho(e_a^x f_b^y).$$

In this sense, the $e_a^x, f_b^y \in C^*(\Gamma \times \Gamma)$ are **universal observables**: only the state needs to be varied. The dual theorem is this:

Theorem

Let $C_{a,b}^{x,y} \in \mathbb{R}_{\geq 0}$ be some coefficients. Then the maximal quantum value of $\sum_{a,b,x,y} C_{a,b}^{x,y} P(a, b|x, y)$ is

$$\left\| \sum_{a,b,x,y} C_{a,b}^{x,y} e_a^x f_b^y \right\|_{C^*(\Gamma \times \Gamma)}.$$

Application: Quantum correlations in the CHSH scenario

- The CHSH scenario is defined by $k = m = 2$ (two binary measurements per party).
- The corresponding group is $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$, which is known to be isomorphic to $\Gamma \cong \mathbb{Z} \rtimes \mathbb{Z}_2$.
- The irreducible representations of such a semidirect product are well-understood. In this case, they are all 2-dimensional.
- Then by the theorem, all quantum correlations in the CHSH scenario can be attained with a qubit for each party.

Application: Tsirelson's Problem

- Let $\mathcal{Q}_c(\Gamma)$ be the set of quantum correlations with the commutativity assumption. Our theorem implies that $\mathcal{Q}_c(\Gamma)$ is closed and convex.
- Let $\mathcal{Q}_\otimes(\Gamma)$ be the set of quantum correlations in Γ with the standard tensor product assumption. An analogous theorem describes the closure[1] $\overline{\mathcal{Q}_\otimes(\Gamma)}$ in terms of $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ instead of $C^*(\Gamma \times \Gamma)$.
- For us, **Tsirelson's problem** asks whether $\mathcal{Q}_c(\Gamma) = \overline{\mathcal{Q}_\otimes(\Gamma)}$.
- **QWEP conjecture** (Kirchberg 1993):

$$C^*(\Gamma \times \Gamma) \stackrel{?}{=} C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$$

Corollary

If the QWEP conjecture is true, then $\mathcal{Q}_c(\Gamma) = \overline{\mathcal{Q}_\otimes(\Gamma)}$ for all Γ.

- A different version of Tsirelson's problem—involving steering of a third system—is equivalent to QWEP.

[1] Deciding whether $\mathcal{Q}_\otimes(\Gamma)$ is already closed seems to be an open problem.

Application: Hierarchy of semidefinite programs I

- By the theorem, $P(a, b|x, y)$ is quantum iff there exists a positive linear map $\rho : C^*(\Gamma \times \Gamma) \to \mathbb{C}$ with

$$P(a, b|x, y) = \rho(e_a^x f_b^y).$$

- Let $L_n \subset C^*(\Gamma \times \Gamma)$ be the linear span of products of up to n generators e_a^x or f_b^y. Then $(L_n)_{n \in \mathbb{N}}$ is an increasing sequence of subspaces with dense union.
- If $P(a, b|x, y)$ is quantum, then

$$s_n : L_n \times L_n \longrightarrow \mathbb{C}, \quad s_n(\gamma_1, \gamma_2) = \rho(\gamma_1^* \gamma_2)$$

  defines a sesquilinear form satisfying $s_n(\gamma_1, \gamma_2) = s_n(\gamma_1', \gamma_2')$ if $\gamma_1^* \gamma_2 = \gamma_1'^* \gamma_2'$ and $s_n(e_a^x, f_b^y)$.

Application: Hierarchy of semidefinite programs II

- If $P(a,b|x,y)$ is quantum, then

$$s_n : L_n \times L_n \longrightarrow \mathbb{C}, \quad s_n(\gamma_1,\gamma_2) = \rho(\gamma_1^*\gamma_2)$$

  defines a sesquilinear form satisfying $s_n(\gamma_1,\gamma_2) = s_n(\gamma_1',\gamma_2')$ if $\gamma_1^*\gamma_2 = \gamma_1'^*\gamma_2'$ and $s_n(e_a^x, f_b^y)$.
- For fixed n, determining whether such an s_n exists is a semidefinite programming problem.
- If $P(a,b|x,y)$ is quantum, then each of these countably many semidefinite programs is feasible.
- The converse follows from the noncommutative Positivstellensatz

$$C^*(\Gamma \times \Gamma)_{\geq 0} = \overline{\{\gamma^*\gamma,\ \gamma \in \cup_n L_n\}},$$

  and a compactness argument.
- This is the **hierarchy of semidefinite programs characterizing quantum correlations** due to Navascués, Pironio and Acín.