Abstract Convexity

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1 Introduction

The material presented in this talk is contained in [Fri09], [Fri].

Historically, the paradigm shifts

$$\text{concrete submanifolds of } \mathbb{R}^n \longrightarrow \text{abstract manifolds}$$

$$\text{concrete sets of symmetries} \longrightarrow \text{abstract groups}$$

were very important for the development of mathematics. As another part of mathematics, convex sets are defined as particular classes of subsets of vector spaces, therefore they belong on the left side. This raises the question:

Question 1.1. Does the concept of **convexity** also admit an abstract counterpart?

This question is answered in the positive by the theory of convex spaces. This theory

- unifies aspects of convex geometry and functional analysis (Banach spaces) with aspects of order theory (semilattices).
- reveals a rich intrinsic geometry of convex sets.
- provides a nice category for the study of convex sets (symmetric monoidal closed, complete and cocomplete).

Disclaimer: Almost none of this material is original research. Convex spaces have been discovered and investigated several times, including the works by

- von Neumann, Morgenstern [vNM07] in the context of economics,
- Stone [Sto49],

- Gudder [Gud79] in the context of quantum mechanics,
- Świrszcz [Świ].
- Flood [Flo81],

Overview: After defining convex spaces, two classes of convex spaces (geometrical, combinatorial) are studied in some detail. Finally, we consider some of the intrinsic geometry of convex space in terms of an intrinsic notion of distance and a separation theorem of Hahn-Banach type.

2 Definition of convex spaces

Definition 2.1. *A* ***convex space*** *is a set C together with a family $(cc_\lambda)_{\lambda\in(0,1)}$ of binary operations*

$$cc_\lambda : C \times C \longrightarrow C, \quad \lambda \in (0,1)$$

satisfying the conditions

- *idempotency:*
$$cc_\lambda(x,x) = x \tag{1}$$
- *parametric commutativity:*
$$cc_\lambda(x,y) = cc_{1-\lambda}(y,x) \tag{2}$$
- *deformed parametric associativity:*
$$cc_\lambda(cc_\mu(x,y),z) = cc_{\widetilde{\lambda}}(x, cc_{\widetilde{\mu}}(y,z)) \tag{3}$$
with
$$\widetilde{\lambda} = \lambda\mu, \qquad \widetilde{\mu} = \begin{cases} \frac{\lambda(1-\mu)}{1-\lambda\mu} & \text{if } \lambda\mu \neq 1 \\ \text{arbitrary} & \text{if } \lambda = \mu = 1. \end{cases}$$

It is known that the vector space axioms imply all equational properties that one expects linear combinations to have. The same holds for convex spaces and convex combinations:

Theorem 2.2. *The relations (1)-(3) are a generating set of all relations that convex combinations in vector spaces have. In other words, an equation between convex combinations of free variables is universally valid if and only if it follows from (1)-(3).*

Proof. Category-theoretic universal algebra. □

Due to this result, it is convenient to replace the notation cc_λ by the usual notation for convex combinations:

$$\lambda x + (1-\lambda) y \equiv cc_\lambda(x,y)$$

3 Convex spaces of geometric type

The obvious prime example of a convex space is a convex subset of a real vector space. For that reason, a convex space is said to be of **geometric type** whenever it can be written as a convex subset of a real vector space.

Theorem 3.1. *A convex space C is of geometric type if and only if the cancellation condition*

$$\lambda x + (1-\lambda)y = \lambda x' + (1-\lambda)y \quad \Longrightarrow \quad x = x' \tag{4}$$

holds for all $x, x', y \in \mathcal{C}$.

Example 3.2. Let $(E, ||\cdot||)$ be a normed vector space. Then the unit ball

$$B_1 \equiv \{x \in E \mid ||x|| \leq 1\}$$

is a convex space in E. Conversely, the convex space B_1 determines the norm via

$$||x|| = \frac{1}{\sup\{r \in \mathbb{R}_{>0} \mid rx \in B_1\}}.$$

Therefore, two mathematical structures (vector space, norm) can be subsumed by a single one (convex space).

Example 3.3. Given two polytopes $P, Q \subseteq \mathbb{R}^n$, their convex combinations can be defined in analogy with Minkowski sums as the new polytope

$$\lambda P + (1-\lambda)Q \equiv \{\lambda x + (1-\lambda)y,\ x \in P,\ y \in Q\}.$$

Therefore, the set of polytopes in $\mathbb{R}^n$ becomes a convex space in its own right. Since the cancellation law (4) holds for polytopes[1], the convex space of polytopes can be embedded into a vector space.

4 Convex spaces of combinatorial type

A different extreme case occurs when the cancellation condition (4) is violated so much that all convex combinations are independent of the weight λ.

Definition 4.1. *A convex space C is said to be of* ***combinatorial type*** *whenever all convex combinations*

$$\lambda x + (1-\lambda)y$$

are independent of λ.

[1] Sketch of proof: suppose that $\lambda P + (1-\lambda)Q = \lambda P' + (1-\lambda)Q$. This implies $n\lambda P + (1-\lambda)Q = n\lambda P' + (1-\lambda)Q$ for any $n \in \mathbb{N}$. Then $P = P'$ follows by choosing n large enough.

Therefore, a convex space of combinatorial is nothing but a set C together with a single binary operation

$$\wedge : C \times C \longrightarrow C$$

which is idempotent by (1), commutative by (2) and associative by (3). By defining an order structure as

$$x \leq y \quad \Leftrightarrow \quad x = x \wedge y$$

it can be directly verified that such a C is nothing but a meet-semilattice, i.e. a partially ordered set such that each pair of elements has a greatest lower bound. (Using meets instead of joins is purely conventional.)

Definition 4.2. *Given any convex space C, a **face** of C is a convex subset $D \subseteq C$ that is extremal in the sense that*

$$\lambda x + (1-\lambda)y \in D \quad \Longrightarrow \quad x \in D,\ y \in D$$

Example 4.3. Let $F = \{0,1\}$ be the meet-semilattice defined by

$$0 \wedge 1 = 0.$$

As a partially ordered set, $0 \leq 1$. The faces of $\{0,1\}$ are $\emptyset$, $\{1\}$ and $\{0,1\}$.

$\{1\} \subseteq \{0,1\}$ is the universal face in the following sense: given any other convex space C, a subset $D \subseteq C$ is a face of C if and only if it has the form $d^{-1}(\{1\})$ for some convex map $d : C \to \{0,1\}$. This is analogous to the specification of a subset $B \subseteq A$ of some set A via its characteristic function $b : A \to \{0,1\}$ as $B = b^{-1}(\{1\})$.

5 Convex spaces of mixed type

A generic convex space is neither of geometric type nor of combinatorial type. Nonetheless, the following classification result shows that every convex space can be decomposed into a combinatorial part and a geometrical part.

Theorem 5.1 (classification, informal version)**.** *Every convex space C can be written as a bundle*

$$C \xrightarrow{p} \widetilde{C}$$

where $\widetilde{C}$ is of combinatorial type and all the fibers $p^{-1}(c)$ for $c \in \widetilde{C}$ are of geometric type.

6 The intrinsic geometry of convex spaces

Convex spaces have a surprisingly rich intrinsic geometric structure. Two facets of this structure are considered here: an intrinsic notion of distance and a separation theorem of Hahn-Banach type.

Definition 6.1. *Given a convex space C of geometric type, define a distance function $d : C \times C \to \mathbb{R}_{\geq 0}$ by*

$$d_C(x, y) \equiv \sup \{|f(x) - f(y)|, \ f : C \to [0, 1] \ \text{convex}\}$$

Under suitable boundedness assumptions on C, this distance function is actually a metric. In particular, the triangle inequality is straightforward to check.

Example 6.2. As an example of how much information this metric can contain, consider the convex set of positive matrices in $M_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ that are of unit trace:

$$C = \{A \in M_n(\mathbb{K}) \mid A \geq 0, \ \mathrm{tr}(A) = 1\}.$$

These sets are important in physics as state spaces of quantum systems. If v is a unit vector, the projection onto v, denoted by P_v, lies in C. Given unit vectors v and w, their distance within C can be calculated to be

$$\sqrt{1 - |\langle v, w\rangle|^2} = d_C(P_v, P_w)$$

Therefore, the structure of C as a convex space contains information about the scalar product in $\mathbb{R}^n$. This works the same way for any Hilbert space instead of $\mathbb{R}^n$.

The next theorem mentions convex subsets with convex complement. For any convex subset of $\mathbb{R}^n$, a convex subset with convex complement defines a half-space in that set. Therefore one can think of convex subsets with convex complement as a generalization of half-spaces.

Theorem 6.3 (Hahn-Banach separation, fig. 1)**.** *Given any convex space C and two convex subspaces $D_1, D_2 \subseteq C$ that are disjoint,*

$$D_1 \cap D_2 = \emptyset,$$

there always exists a convex subspace S such that its complement is also convex and the inclusions

$$D_1 \subseteq S, \quad D_2 \subseteq C \setminus S$$

hold.

Idea of proof: consider pairs of subspaces (E_1, E_2) such that $D_i \subseteq E_i$ and $E_1 \cap E_2 = \emptyset$. The set of all such pairs is non-empty, and can be ordered by inclusion. Every ascending chain has an upper bound given by taking the union of the whole chain for each component. Therefore by Zorn's lemma, there exists a pair (E_1, E_2) having these properties which is maximal with respect to inclusion. A geometric argument shows that this pair cannot be maximal if $E_1 \cup E_2 \neq C$. But since the pair is assumed to be maximal, it follows that $E_1 \cup E_2 = C$, so that one can set $S = E_1$, which contains D_1. Then the complement $C \setminus S = E_2$ is automatically also convex and contains D_2.

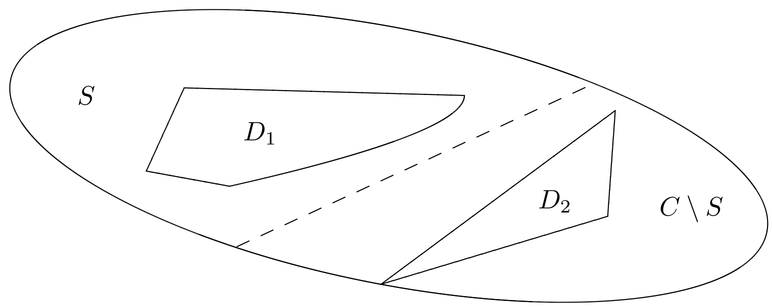

Figure 1: Illustration of theorem 6.3. The dashed line itself may belong to $\mathcal{S}$ completely, not at all, or just for some part.

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