The Geometry of Black Holes

Introduction

What is the gravitational field of a point mass M? In Newtonian Gravity:

$$F = G\frac{mM}{r^2} \tag{1}$$

General Relativity (Einstein 1915): Gravity is curvature of spacetime.

So what is the General Relativity analogue? This problem was solved by Karl Schwarzschild (1916): *Schwarzschild geometry*. As already conjectured by John Michell (1783), an *event horizon* is formed: When inside the horizon, even light is too slow to escape! An object with a horizon like this is called a **black hole**.

However, in reality point masses do not exist; most objects are not compact enough to form a black hole. But also, the Schwarzschild solution describes the exterior gravitational field of any spherically symmetric body, for example a star, so it is widely useful.

Overview

Goal: Understand the Schwarzschild gravitational geometry and introduce General Relativity along the way.

- From Special Relativity to Lorentzian manifolds
- Einstein's equations and the Schwarzschild solution
- The Kruskal extension
- Outlook: Astrophysics of black holes

Differential Geometry Notation

$\alpha, \beta, \gamma, \ldots$ are tensor indices

Einstein summation convention: e.g. $S^{\alpha\beta}{}_{\gamma} T_{\beta\delta} \equiv \sum_{\beta} S^{\alpha\beta}{}_{\gamma} T_{\beta\delta}$

Index raising and lowering: e.g. $S_{\alpha} \equiv g_{\alpha\beta} S^{\alpha}$

Levi-Civita connection: Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$

Curvature:

Riemann curvature tensor: $R_{\alpha\beta\gamma}{}^{\delta}$

Ricci tensor: $R_{\alpha\gamma} \equiv R_{\alpha\beta\gamma}{}^{\beta}$

Curvature scalar: $R \equiv R_{\alpha}{}^{\alpha}$

Special Relativity: Kinematics

Minkowski space $M \cong \mathbb{R}^4$ with symmetric bilinear form $\eta_{\alpha\beta}$ of signature $(+,-,-,-)$. In an orthonormal basis, $\eta_{\alpha\beta} = \mathrm{diag}(+1,-1,-1,-1)$, so

$$\eta_{\alpha\beta}\, x^\alpha y^\alpha = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \tag{2}$$

We think of (x^0, x^1, x^2, x^3) as $(\mathrm{time}, \mathrm{space}, \mathrm{space}, \mathrm{space})$. However, the splitting into time and space directions is not unique!

An observer is described by a *worldline* $x^\alpha(t) = (x^0(t), x^1(t), x^2(t), x^3(t))$ with arbitrary parametrization. The elapsed time on his watch is

$$d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \quad \text{or} \quad \Delta\tau = \int_{t_1}^{t_2} \sqrt{\eta_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta}\, dt \tag{3}$$

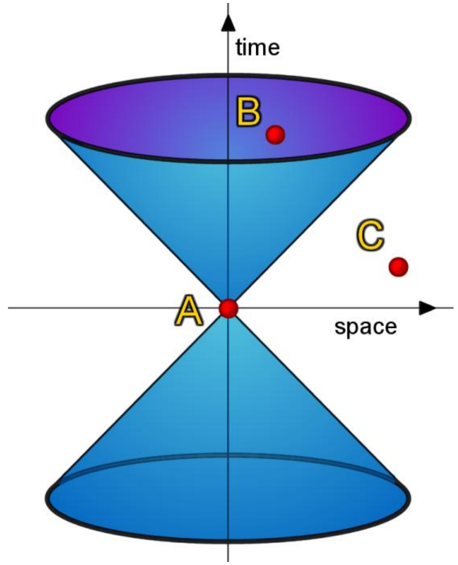
Special Relativity: Dynamics

An observer upon which no force acts does not accelerate, so his worldline is a straight line. By choosing an *affine parametrization*, we can achieve

$$\ddot{x}^\alpha(t) = 0 \tag{4}$$

Furthermore, we have $\dot{x}^\alpha \dot{x}_\alpha = \eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$ for light rays; drawing the collection of all these *null vectors* gives the *light cone*.

For observers or massive particles, we can scale the affine parameter such that $\dot{x}^\alpha \dot{x}_\alpha = 1$.



General Relativity I: Lorentzian manifolds

Immediate generalization: consider now *any* 4-manifold with a pseudo-Riemannian metric $g_{\alpha\beta}$ of signature $(+,-,-,-)$.

Moving (massive) objects or light rays are still described by worldlines $x^\alpha(t)$. Measured time of an observer is given by

$$d\tau^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \tag{5}$$

Their equation of motion now is given by them moving on *geodesics*

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\, \dot{x}^\beta \dot{x}^\gamma = 0 \tag{6}$$

which takes into account the gravitational field via $\Gamma^\alpha_{\beta\gamma}$.

Schwarzschild Geometry I: Symmetries

Determining the field of a point mass means finding a Lorentzian manifold $\mathcal{M}$ with the symmetries:
- rotation invariance: an $SO(3)$ action with S^2 orbits
- time translation invariance: an $\mathbb{R}$ action with $\mathbb{R}$ orbits
- time reflection symmetry: a $\mathbb{Z}_2$ action

which all commute with each other.

These give very powerful constraints on the geometry. Using Frobenius' theorem, one can show that the metric is of the form

$$d\tau^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = f(r)\, dt^2 - h(r)\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (7)$$

where θ and ϕ are spherical coordinates on the S^2 orbits and t parametrizes the time translation orbits. r is such that the vector field ∂_r is orthogonal to the other coordinate fields and the area of an S^2 orbit is $4\pi r^2$.

General Relativity II: Einstein's Equations

Given some matter distribution described by a *stress-energy tensor* $T_{\alpha\beta}$, the spacetime geometry has to satisfy the *Einstein field equations*

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta} \tag{8}$$

On the other hand, the spacetime geometry $g_{\alpha\beta}$ will influence the flow of matter and thus $T_{\alpha\beta}$. This complicated feedback is what makes explicit solutions in general relativity extremely hard to find; only few are known.

Usually, the topology of the spacetime is not known beforehand, but determined *after* solving for the metric, in such a way that geodesics extend as far as possible.

Schwarzschild Geometry II: Solution of Einstein's Equations

In our case, $T_{\alpha\beta} = 0$, since we have to exclude the point mass from our coordinate chart. Then the Einstein equations reduce to $R_{\alpha\beta} = 0$. Solving this for f and h, we get the general solution, with some constant of integration C,

$$d\tau^2 = (1 - \frac{C}{r})\, dt^2 - (1 - \frac{C}{r})^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{9}$$

For $r \to \infty$ we get the Minkowski metric back, which indicates that the gravitational field abates. For $r \to 0$ and $r \to C$, our coordinates have singularities. As we will find out, $r = 0$ is a real singularity of the geometry, while at $r = C$ we only have a coordinate singularity.

Schwarzschild Geometry III: Interpreting the solution

Imagine a small satellite initially resting at a position $r \gg C$; note that resting means $\dot{x}^\alpha = (1, 0, 0, 0)$. It will start accelerating according to

$$\ddot{r} = -\Gamma^r_{tt} = -\frac{C(r-C)}{2r^3} \approx -\frac{C}{2r^2} \tag{10}$$

Since the field is weak for large r, we are allowed to use the Newtonian approximation

$$\ddot{r} = -\frac{M}{r^2} \tag{11}$$

We can identify $C = 2M$ as parametrizing the $mass$ of our gravitating object!

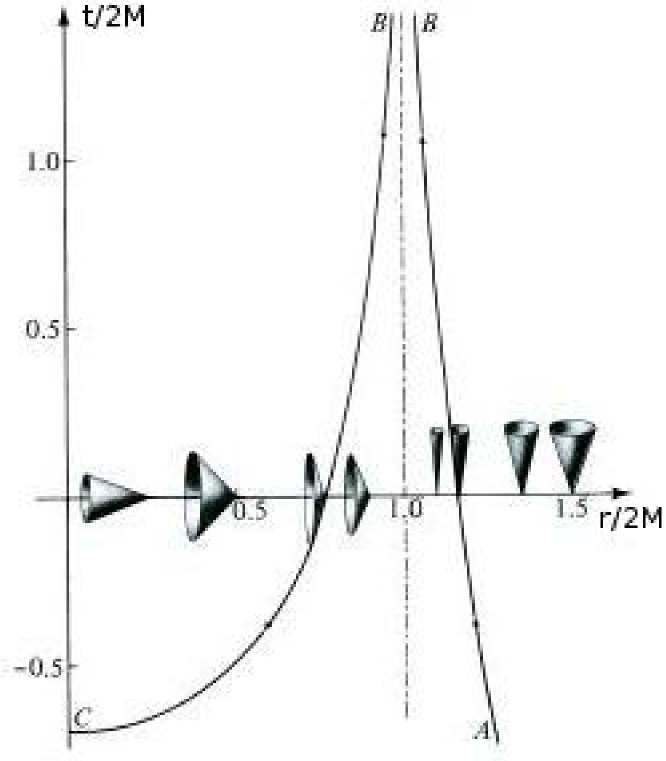
$$d\tau^2 = (1 - \frac{2M}{r})\, dt^2 - (1 - \frac{2M}{r})^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{12}$$

Schwarzschild Geometry IV: Light rays

At every spacetime point, we have a light cone sitting in its tangent space.
The radial (that is, $d\theta = d\phi = 0$) light rays are described by

$$\frac{dt}{dr} = \pm\frac{r}{r-2M} \tag{13}$$

However, it turns out that, in affine parametrization, $\ddot{r} = 0$ for a light ray - an incoming one goes to $t = \infty$ while $r \to 2M$, "comes back" and hits $r = 0$ in finite affine parameter. But a distant observer **never** sees the light ray leave the $r > 2M$ region!



Schwarzschild Geometry V: the horizon

By time inversion symmetry, there exist also light rays starting at $r = 0$, heading off to $t = -\infty$ at $r = 2M$, and finally escaping to $r \to \infty, t \to \infty$.

The future half of a light cone for $r > 2M$ corresponds to the inpointing half of a light cone in the interior region $r < 2M$. In the same way that we can only send signals along a future light cone in ordinary space, we will only be able to send signals in the direction of decreasing r when we are at $r < 2M$. In particular, we cannot send any signal out to the $r > 2M$ region.

Consequently, $r = 2M$ is a $horizon$ for light rays.

Worldlines for infalling massive objects look similar. As seen from a distant observer, an infalling object also never crosses the horizon, only gets closer and closer.

Schwarzschild Geometry VI: more General Relativity effects

Observable physical consequences of the Schwarzschild solution abounding in the $r > 2M$ region and their values for the solar system:

- Perihelion advance: $43''/\mathrm{yr}$ for Mercury's orbit
- Gravitational redshift of light: $2 \cdot 10^{-3}$ for the sun
- Gravitational bending and lensing of light: $1.8''$ along the surface of the sun

The Kruskal Extension I: Coordinate Transformation

The previous considerations suggest that $r = 2M$ might be a coordinate singularity that disappears under an appropriate coordinate transformation. In 1960, Martin Kruskal found the following answer: consider coordinates (T, X, θ, ϕ) implicitly defined by

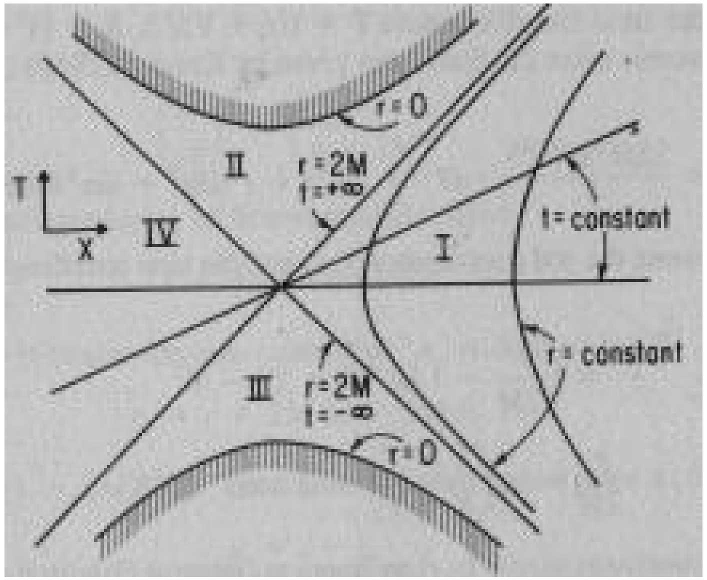
$$T^2 - X^2 = \left(1 - \frac{r}{2M}\right) e^{r/2M}, \quad \frac{T}{X} = \tanh\left(\frac{t}{4M}\right) \tag{14}$$

Given (T, X), there is a unique solution for (t, r) provided that $T^2 - X^2 < 1$. Thinking of $r = r(T, X)$ as a function implicitly defined by the first equation, the metric takes on the form

$$d\tau^2 = \frac{32M^3 e^{-r/2M}}{r}(dT^2 - dX^2) - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{15}$$

The Kruskal Extension II: a white hole!

Now radial light rays are given by $dT = \pm dX$, so they are just diagonal lines. The horizon lies at $T = \pm X$ and is perfectly regular. However, note that T and X are only defined up to sign from t and r: up to some critical points, the Kruskal extension is a *double cover* of the Schwarzschild geometry!

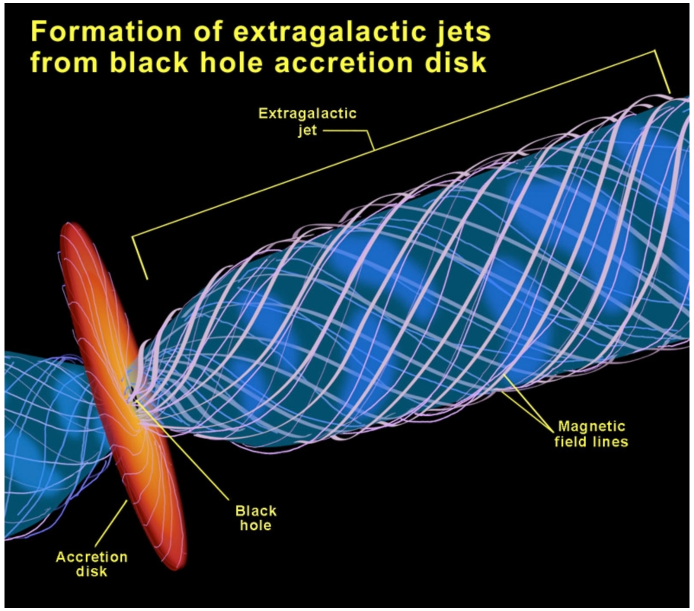


Obviously, any light ray or massive object not staying on the horizon will either finally end up in the singularity $r = 0$ in region II - or be coming from region III and the $r = 0$ singularity there. In this sense, region III is referred to as a *white hole*: it is not possible to get there, only to come from there. It is the time-reversed part of region II.

Observations of black holes

Formation predicted by general relativity for stars of $\gtrsim 4$ solar masses due to collapsing at the end of their life cycle.

Black holes should be very visible! Indeed, accretion disks of different sizes are observed that are probably due to black holes. Nowadays there is evidence that many galaxies have a supermassive black hole in their center, including our Milky Way!



With some luck, gravitational wave detectors like LIGO will detect a few black hole coalescences in the next few years: though merging black hole events are thought to be very rare, they should create huge amounts of gravitational waves in an unambiguous pattern.

References

[1] Robert Wald: *General Relativity*, Chicago Press 1984.

[2] Charles Misner, Kip Thorne, John Wheeler: *Gravitation*, W. H. Freeman 1973. Also known as "the bible of gravitational physics"!

[3] Hans Ohanian: *Gravitation and Spacetime*, W. W. Norton & Company Inc., 1976.

[4] Roger Penrose: *The Road to Reality*, Knopf 2005.