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SYNCHRONIZATION OF DISCRETE-TIME DYNAMICAL NETWORKS WITH TIME-VARYING COUPLINGS*

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Abstract. We study the local complete synchronization of discrete-time dynamical networks with time-varying couplings. Our conditions for the temporal variation of the couplings are rather general and include variations in both the network structure and the reaction dynamics; the reactions could, for example, be driven by a random dynamical system. A basic tool is the concept of the Hajnal diameter, which we extend to infinite Jacobian matrix sequences. The Hajnal diameter can be used to verify synchronization, and we show that it is equivalent to other quantities which have been extended to time-varying cases, such as the projection radius, projection Lyapunov exponents, and transverse Lyapunov exponents. Furthermore, these results are used to investigate the synchronization problem in coupled map networks with time-varying topologies and possibly directed and weighted edges. In this case, the Hajnal diameter of the infinite coupling matrices can be used to measure the synchronizability of the network process. As we show, the network is capable of synchronizing some chaotic map if and only if there exists an integer $T > 0$ such that for any time interval of length T, there exists a vertex which can access other vertices by directed paths in that time interval.

Key words. synchronization, dynamical networks, time-varying coupling, Hajnal diameter, projection joint spectral radius, Lyapunov exponents, spanning tree

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1. Introduction.

Synchronization of dynamical processes on networks is presently an active research topic. It represents a mathematical framework that on the one hand can elucidate—desired or undesired—synchronization phenomena in diverse applications. On the other hand, the synchronization paradigm is formulated in such a manner that powerful mathematical techniques from dynamical systems and graph theory can be utilized. A standard version is

$$x^i(t+1) = f^i(x^1(t), x^2(t), \dots, x^m(t)), \quad i = 1, 2, \dots, m, \tag{1.1}$$

where $t \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ denotes the discrete time, $x^i(t) \in \mathbb{R}$ denotes the state variable of unit (vertex) i, and for $i = 1, 2, \dots, m$, $f^i : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function. This dynamical systems formulation contains two aspects. One of them is the reaction dynamics at each node or vertex of the network. The other one is the coupling structure, that is, whether and how strongly the dynamics at one node is directly influenced by the states of the other nodes.

Equation (1.1) clearly is an abstraction and simplification of synchronization problems found in applications. On the basis of understanding the dynamics of (1.1), research should then move on to more realistic scenarios. Therefore, in the present work, we address the question of synchronization when the right-hand side of (1.1) is allowed to vary in time. Thus, not only the dynamics itself is a temporal process, but

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also the underlying structure changes in time, albeit in some applications that may occur on a slower time scale.

The essence of the hypotheses on $f = [f^1, \dots, f^m]$ needed for synchronization results (to be stated in precise terms shortly) is that synchronization is possible as an invariant state, that is, when the dynamics starts on the diagonal $[x, \dots, x]$, it will stay there, and that this diagonal possesses a stable attracting state. The question about synchronization then is whether this state is also attracting for dynamical states $[x^1, \dots, x^m]$ outside the diagonal, at least locally, that is, when the components x^i are not necessarily equal but close to each other. This can be translated into a question about transverse Lyapunov exponents, and one typically concludes that the existence of a synchronized attractor in the sense of Milnor. In our contribution, we can already strengthen this result by concluding (under appropriate assumptions) the existence of a synchronized attractor in the strong sense instead of only in the weaker sense of Milnor. (We shall call this local complete synchronization.) This comes about because we achieve a reformulation of the synchronization problem in terms of Hajnal diameters (a concept to be explained below).

Our work, however, goes beyond that. As already indicated, our main contribution is that we can study the local complete synchronization of general coupled networks with time-varying coupling functions, in which each unit is dynamically evolving according to

$$x^i(t+1) = f_t^i(x^1(t), x^2(t), \dots, x^m(t)), \quad i = 1, 2, \dots, m. \tag{1.2}$$

This formulation, in fact, covers both aspects described above, the reaction dynamics as well as the coupling structure. The main purpose of the present paper then is to identify general conditions under which we can prove synchronization of the dynamics (1.2). Thus, we can handle variations of the reaction dynamics as well as those of the underlying network topology. We shall mention below various applications where this is of interest.

Before that, however, we state our technical hypothesis on the right-hand side of (1.2): for each $t \in \mathbb{Z}^+$, $f_t^i : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function with the following hypothesis.

(H$_1$) *There exists a C^1 function $f(s) : \mathbb{R} \to \mathbb{R}$ such that*

$$f_t^i(s, s, \dots, s) = f(s)$$

holds for all $s \in \mathbb{R}$, $t \in \mathbb{Z}^+$, and $i = 1, 2, \dots, m$. Moreover, for any compact set $K \subset \mathbb{R}^m$, f_t^i and the Jacobian matrices $[\partial f_t^i / \partial x^j]_{i,j=1}^m$ are all equicontinuous in K with respect to $t \in \mathbb{Z}^+$ and the latter are all nonsingular in K.

This hypothesis ensures that the diagonal synchronization manifold

$$\mathcal{S} = \left\{ [x^1, x^2, \dots, x^m]^\top \in \mathbb{R}^m : \ x^i = x^j, \ i, j = 1, 2, \dots, m \right\}$$

is an invariant manifold for the evolution (1.2). If $x^1(t) = x^2(t) = \cdots = x^m(t) = s(t)$ denotes the synchronized state, then

$$s(t+1) = f(s(t)). \tag{1.3}$$

For the synchronized state (1.3), we assume the existence of an attractor, as follows.

(H$_2$) *There exists a compact asymptotically stable attractor A for* (1.3). *That is*, (i) *$A \subset \mathbb{R}$ is a forward invariant set*; (ii) *for any neighborhood U of A there* *exists a neighborhood V of A such that $f^n(V) \subset U$ for all $n \in \mathbb{Z}^+$;* (iii) *for any sufficiently small neighborhood U of A, $f^n(U)$ converges to A, in the sense that for any neighborhood V, there exists n_0 such that $f^n(U) \subset V$ for $n \geq n_0$;* (iv) *there exists $s^* \in A$ for which the ω-limit set is A.*

Let A^m denote the Cartesian product $A \times \cdots \times A$ (m times). Local complete synchronization (synchronization for simplicity) is defined in the sense that the set $\mathcal{S} \cap A^m = \{[x, \ldots, x] : x \in A\}$ is an asymptotically stable attractor in $\mathbb{R}^m$. That is, for the coupled dynamical system (1.2), differences between components converge to zero if the initial states are picked sufficiently near $\mathcal{S} \cap A^m$, i.e., if the components are all close to the attractor A and if their differences are sufficiently small. In order to show such a synchronization, one needs a third hypothesis (H_3) that in technical terms is about Lyapunov exponents transverse to the diagonal. That is, while the dynamics on the attractor may well be expanding (the attractor might be chaotic), the transverse directions need to be suitably contracting to ensure synchronization. The corresponding hypothesis (H_3) will be stated below (see (3.2)) because it requires the introduction of crucial technical concepts.

It is an important aspect of our work that we shall derive the attractivity here in the classical sense, and not in the sense of Milnor, i.e., not only some set of positive measure, but a full neighborhood is attracted. For details about the difference between Milnor attractors and asymptotically stable attractors, see [1, 2]. Usually, when studying synchronization, one derives only the existence of a Milnor attractor; see [3].

The motivation for studying (1.2) comes from the well-known coupled map lattices (CML) [4], which can be written as follows:

$$x^i(t+1) = f(x^i(t)) + \sum_{j=1}^{m} L_{ij} f(x^j(t)), \; i = 1, 2, \ldots, m, \tag{1.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable map and $L = [L_{ij}]_{i,j=1}^m \in \mathbb{R}^{m\times m}$ is the diffusion matrix, which is determined by the topological structure of the network and satisfies $L_{ij} \geq 0$ for all $i \neq j$, and $\sum_{j=1}^m L_{ij} = 0$ for all $i = 1, 2, \ldots, m$. Letting $x = [x^1, x^2, \ldots, x^m]^\top \in \mathbb{R}^m$, $F(x) = [f(x^1), f(x^2), \ldots, f(x^m)]^\top \in \mathbb{R}^m$, and $G = I_m + L$, where I_m denotes the identity matrix of dimension m, the CML (1.4) can be written in the matrix form

$$x(t+1) = GF(x(t)), \tag{1.5}$$

where $G = [G_{ij}]_{i,j=1}^m \in \mathbb{R}^{m\times m}$ denotes the coupling and satisfies $G_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^m G_{ij} = 1$ for all $i = 1, 2, \ldots, m$. So, if $G_{ii} \geq 0$ holds for all $i = 1, 2, \ldots, m$, then G is a stochastic matrix.

Recently, synchronization of CML has attracted increasing attention [3, 5, 6, 7, 8]. Linear stability analysis of the synchronization manifold was proposed and transverse Lyapunov exponents were used to analyze the influence of the topological structure of networks. In [1], conditions for generalized transverse stability were presented. If the transverse (normal) Lyapunov exponents are negative, a chaotic attractor on an invariant submanifold can be asymptotically stable over the manifold. In [9, 10] it was shown that chaos synchronization in a network of nonlinear continuous-time or discrete-time dynamical systems, respectively, is possible if and only if the corresponding graph has a spanning tree. However, synchronization analysis has so far been limited to autonomous systems, where the interactions between the vertices (state components) are static and do not vary through time.

In the social, natural, and engineering real world, the topology of the network often varies through time. In communication networks, for example, one must consider dynamical networks of moving agents. Since the agents are moving, some of the existing connections can fail simply due to occurrence of an obstacle between agents. Also, some new connections may be created when one agent enters the effective region of other agents [11]. On top of that, randomness may also occur. In a communication network, the information channel of two agents at each time may be random [12]. When an error occurs at some time, the connections in the system will vary. In [11, 12, 13], synchronization of multiagent networks was considered where the state of each vertex is adapted according to the states of its connected neighbors with switching connecting topologies. This multiagent dynamical network can be written in discrete-time form as

$$x^i(t+1) = \sum_{j=1}^{m} G_{ij}(t)x^j(t),\ i = 1, 2, \dots, m, \tag{1.6}$$

where $x^j(t) \in \mathbb{R}$ is the state variable of vertex j and $[G_{ij}(t)]_{i,j=1}^m$, $t \in \mathbb{Z}^+$, are stochastic matrices. In [14] a convexity-conserving coupling function was considered that is equivalent to the linear coupling function in (1.6). It was found that the connectivity of the switching graphs plays a key role in the synchronization of multiagent networks with switching topologies. Also, in the recent literature [15, 16, 17], synchronization of continuous-time dynamical networks with time-varying topologies was studied. The time-varying couplings investigated, however, are specific, with either symmetry [15], node balance [16], or fixed time average [17].

Therefore, it is natural to investigate the synchronization of CML with general time-varying connections as

$$x(t+1) = G(t)F(x(t)), \tag{1.7}$$

where $G(t) = [G_{ij}(t)]_{i,j=1}^m \in \mathbb{R}^{m\times m}$ denotes the coupling matrix at time t and $F(x) = [f(x_1), \dots, f(x_n)]^\top$ is a differentiable function. We shall address this problem in the context of the general coupled system (1.2).

Let

$$x(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^m(t) \end{bmatrix} \quad \text{and} \quad F_t(x(t)) = \begin{bmatrix} f_t^1(x^1(t), \cdots, x^m(t)) \\ f_t^2(x^1(t), \cdots, x^m(t)) \\ \vdots \\ f_t^m(x^1(t), \cdots, x^m(t)) \end{bmatrix}.$$

Equation (1.2) can be rewritten in matrix form:

$$x(t+1) = F_t(x(t)). \tag{1.8}$$

The time-varying coupling can have a special form and may be driven by some other dynamical system. Let $\mathcal{Y} = \{\Omega, \mathcal{F}, P, \theta^{(t)}\}$ denote a metric dynamical system (MDS), where Ω is the metric state space, $\mathcal{F}$ is the σ-algebra, P is the probability measure, and $\theta^{(t)}$ is a semiflow satisfying $\theta^{(t+s)} = \theta^{(t)} \circ \theta^{(s)}$ and $\theta^{(0)} = \mathrm{id}$, where id denotes the identity map. Then the coupled system can be regarded as a random dynamical system (RDS) driven by $\mathcal{Y}$:

$$x(t+1) = F(x(t), \theta^{(t)}\omega),\ t \in \mathbb{Z}^+,\ \omega \in \Omega. \tag{1.9}$$

In fact, one can regard the dynamical system (1.9) as a skew product semiflow,

$$\Theta : \mathbb{Z}^+ \times \Omega \times \mathbb{R}^m \to \Omega \times \mathbb{R}^m,$$
$$\Theta^{(t)}(\omega, x) = (\theta^{(t)}\omega, x(t)).$$

Furthermore, the coupled system can have the form

$$x(t+1) = F(x(t), u(t)), \ t \in \mathbb{Z}^+, \tag{1.10}$$

where u belongs to some function class $\mathcal{U}$ and may be interpreted as an external input or force. Then, defining $[\theta^{(t)}u](\tau) = u(t+\tau)$ as a shift map, the system (1.10) has the form of (1.9). In this paper, we first investigate the general time-varying case of the system (1.8) and also apply our results to systems of the form (1.9).

To study synchronization of the system (1.8), we use its variational equation by linearizing it. Consider the difference $\delta x^i(t) = x^i(t) - f^{(t-t_0)}(s_0)$. This implies that $\delta x^i(t) - \delta x^j(t) = x^i(t) - x^j(t)$ holds for all $i, j = 1, 2, \ldots, m$. We have

$$\delta x^i(t+t_0) = \sum_{j=1}^{m} \frac{\partial f^i_{t+t_0-1}}{\partial x^j}(f^{(t-1)}(s_0))\delta x^j(t+t_0-1), \ i = 1, 2, \ldots, m, \tag{1.11}$$

where for simplicity we use the notation $\frac{\partial f^i_{t+t_0-1}}{\partial x^j}(f^{(t-1)}(s_0))$ to denote $\frac{\partial f^i_{t+t_0-1}}{\partial x^j}$ $(f^{(t-1)}(s_0), \ldots, f^{(t-1)}(s_0))$. Let

$$\delta x(t) = \begin{bmatrix} \delta x^1(t) \\ \vdots \\ \delta x^m(t) \end{bmatrix}, \quad D_t(s) = \left[\frac{\partial f^i_t}{\partial x^j}(s)\right]_{i,j=1}^m.$$

The variational equation (1.11) is written in matrix form,

$$\delta x(t+t_0) = D_{t+t_0-1}(f^{(t-1)}(s_0))\delta x(t+t_0-1). \tag{1.12}$$

For the Jacobian matrix, the following lemma is an immediate consequence of hypothesis (H_1).

Lemma 1.1.

$$\sum_{j=1}^{m} \frac{\partial f^i_t}{\partial x_j}(s, s, \ldots, s) = f'(s), \quad i = 1, 2, \ldots, m \quad \textit{and} \quad t \in \mathbb{Z}^+.$$

Namely, all rows of the Jacobian matrix $[\partial f^i_t/\partial x_j]^m_{i,j=1}$ evaluated on the synchronization manifold $\mathcal{S}$ have the same sum, which is equal to $f'(s)$.

As a special case, if the time variation is driven by some dynamical system $\mathcal{Y} = \{\Omega, \mathcal{F}, P, \theta^{(t)}\}$, then the variational system does not depend on the initial time t_0, but only on (s_0, ω). Thus, the Jacobian matrix can be written in the form $D(f^{(t)}(s_0), \theta^{(t)}\omega) = D_t(f^{(t)}(s))$, by which the variational system can be written as

$$\delta x(t+1) = D(f^{(t)}(s_0), \theta^{(t)}\omega)\delta x(t). \tag{1.13}$$

In this paper, we first extend the concept of the Hajnal diameter to general matrices. A matrix with Hajnal diameter less than one has the property of compressing the convex hull of $\{x^1, \ldots, x^m\}$. Consequently, for an infinite sequence of time-varying

Jacobian matrices, the average compression rate can be used to verify synchronization. Since the Jacobian matrices have identical row sums, the (skew) projection along the diagonal synchronization direction can be used to define the projection joint spectral radius, which equals the Hajnal diameter. Furthermore, we show that the Hajnal diameter is equal to the largest Lyapunov exponent along directions transverse to the synchronization manifold; hence, it can also be used to determine whether the coupled system (1.2) can be synchronized.

Second, we apply these results to discuss the synchronization of the CML with time-varying couplings. As we shall show, the Hajnal diameter of infinite coupling stochastic matrices can be utilized to measure the synchronizability of the coupling process. More precisely, the coupled system (1.7) synchronizes if the sum of the logarithm of the Hajnal diameter and the largest Lyapunov exponent of the uncoupled system is negative. Using the equivalence of the Hajnal diameter, projection joint spectral radius, and transverse Lyapunov exponents, we study some particular examples for which the Hajnal diameter can be computed, including static coupling, a finite coupling set, and a multiplicative ergodic stochastic matrix process. We also present numerical examples to illustrate our theoretical results.

The connection structure of the CML (1.5) naturally gives rise to a graph, where each unit can be regarded as a vertex. Hence, we associate the coupling matrix G with a graph $\Gamma = (V, E)$, with the vertex set $V = \{1, 2, \dots, m\}$ and the edge set $E = \{e_{ij}\}$, where there exists a directed edge from vertex j to vertex i if and only if $G_{ij} > 0$. The graphs we consider here are assumed to be simple (that is, without loops and multiple edges) but are allowed to be directed and weighted. That is, we do not assume a symmetric coupling scheme.

We extend this idea to an infinite graph sequence $\{\Gamma(t)\}$. That is, we regard a time-varying graph as a graph process $\{\Gamma(t)\}_{t\in\mathbb{Z}^+}$. Define $\Gamma(t) = [V, E(t)]$, where $V = \{1, 2, \dots, m\}$ denotes the vertex set and $E(t) = \{e_{ij}(t)\}$ denotes the edge set of the graph at time t. The time-varying coupling matrix $G(t)$ might then be regarded as a function of the time-varying graph sequence, i.e., $G(t) = G(\Gamma(t))$. A basic problem that arises is determining which kind of sequence can ensure the synchrony of the coupled system for some chaotic synchronized state $s(t + 1) = f(s(t))$. As we shall show, the property that the union of the $\Gamma(t)$ contains a spanning tree is important for synchronizing chaotic maps. We prove that under certain conditions, the coupling graph process can synchronize some chaotic maps if and only if there exists an integer $T > 0$ such that there exists at least one vertex j from which any other vertex can be accessible within a time interval of length T.

This paper is organized as follows. In section 2, we present some definitions and lemmas on the Hajnal diameter, projection joint spectral radius, projection Lyapunov exponents, and transverse Lyapunov exponents for generalized Jacobian matrix sequences as well as stochastic matrix sequences. In section 3, we study the synchronization of the generalized coupled discrete-time systems with time-varying couplings (1.2). In section 4, we discuss the synchronization of the CML with time-varying couplings (1.7) and study the relation between synchronizability and coupling graph process topologies. In addition, we present some examples where synchronizability is analytically computable. In section 5, we present numerical examples to illustrate the theoretical results, and we conclude the paper in section 6.

2. Preliminaries. In this section we present some definitions and lemmas on matrix sequences. First, we extend the definitions of the Hajnal diameter and the projection joint spectral radius, introduced in [18, 19, 20] for stochastic matrices, to generalized time-varying matrix sequence. Furthermore, we extend Lyapunov exponents and projection Lyapunov exponents to the general time-varying case and discuss their relation. Second, we specialize these definitions to stochastic matrix sequences and introduce the relation between a stochastic matrix sequence and graph topology.

2.1. General definitions. We study the generalized time-varying linear system

$$u(t+t_0+1) = L_{t+t_0}(\varrho^{(t)}(\phi))u(t+t_0), \tag{2.1}$$

where $\varrho^{(t)}$ is defined by a random dynamical system $\{\Phi, \mathcal{B}, P, \varrho^{(t)}\}$, where Φ denotes the state space, $\mathcal{B}$ the σ-algebra on Φ, P the probability measure, and $\varrho^{(t)}$ a semiflow. Studying the linear system (2.1) comes from the variational system of the coupled system (1.2). For the variational system (1.12), $\varrho^{(t)}(\cdot)$ represents the synchronized state flow $f^{(t)}(\cdot)$. And, if $L_t(\cdot)$ is independent of t, then the linear system (2.1) can be rewritten as

$$u(t+1) = L(\varrho^{(t)}(\phi))u(t). \tag{2.2}$$

Thus, it can represent the variational system (1.13) as a special case, where $\varrho^{(t)}$ is the product flow $(f^{(t)}(\cdot), \theta^{(t)}(\cdot))$. Hence, the linear system (2.1) can unify the two cases of variational systems (1.12), (1.13) of the coupled system (1.2), (1.9).

For this purpose, we define a generalized matrix sequence map $\mathcal{L}$ from $\mathbb{Z}^+ \times \Phi$ to $2^{\mathbb{R}^{m\times m}}$,

$$\begin{aligned} \mathcal{L} : \mathbb{Z}^+ \times \Phi &\to 2^{\mathbb{R}^{m\times m}} \\ (t_0, \phi) &\mapsto \{L_{t+t_0}(\varrho^{(t)}\phi)\}_{t\in\mathbb{Z}^+}, \end{aligned} \tag{2.3}$$

where $2^{\mathbb{R}^{m\times m}}$ denotes the set containing all subsets of $\mathbb{R}^{m\times m}$. In [18, 19], the concept of the Hajnal diameter was introduced to describe the compression rate of a stochastic matrix. We extend it to general matrices below.

DEFINITION 2.1. *For a matrix L with row vectors $g_1, \ldots, g_m$ and a vector norm $\|\cdot\|$ in $\mathbb{R}^m$, the Hajnal diameter of L is defined by*

$$\operatorname{diam}(L, \|\cdot\|) = \max_{i,j} \|g_i - g_j\|.$$

We also introduce the Hajnal diameter for a matrix sequence map $\mathcal{L}$.

DEFINITION 2.2. *For a generalized matrix sequence map $\mathcal{L}$, the Hajnal diameter of $\mathcal{L}$ at $\phi \in \Phi$ is defined by*

$$\operatorname{diam}(\mathcal{L}, \phi) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \left\{ \operatorname{diam}\left(\prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) \right) \right\}^{\frac{1}{t}},$$

where $\prod$ denotes the left matrix product: $\prod_{k=1}^n A_k = A_n \times A_{n-1} \times \cdots \times A_1$.

The Hajnal diameter for the infinite matrix sequence map $\mathcal{L}$ does not depend on the choice of the norm. In fact, all norms in a Euclidean space are equivalent and any additional factor is eliminated by the power $1/t$ and the limit as $t \to \infty$.

Let $\mathcal{H} \subset \mathbb{R}^{m\times m}$ be a class of matrices having the property that all row sums are the same. Thus, all matrices in $\mathcal{H}$ share the common eigenvector $e_0 = [1, 1, \ldots, 1]^\top$, where the corresponding eigenvalue is the row sum of the matrix. Then the projection joint spectral radius can be defined for a generalized matrix sequence map $\mathcal{L}$, similar to that introduced in [20] as follows.

DEFINITION 2.3. *Suppose $\mathcal{L}(t_0,\phi)\subset\mathcal{H}$ for $t_0\in\mathbb{Z}^+$ and $\phi\in\Phi$. Let $\mathcal{E}_0$ be the subspace spanned by the synchronization direction $e_0=[1,1,\dots,1]^\top$, and let P be any $(m-1)\times m$ matrix with exact kernel $\mathcal{E}_0$. We denote by $\hat{L}\in\mathbb{R}^{(m-1)\times(m-1)}$ the (skew) projection of matrix $L\in\mathcal{H}$ as the unique solution of*

$$PL=\hat{L}P. \tag{2.4}$$

The projection joint spectral radius of the generalized matrix sequence map $\mathcal{L}$ is defined as

$$\hat{\rho}(\mathcal{L},\phi)=\overline{\lim_{t\to\infty}}\sup_{t_0\geq 0}\left\|\prod_{k=t_0}^{t_0+t-1}\hat{L}_k(\varrho^{(k-t_0)}\phi)\right\|^{\frac{1}{t}}.$$

One can see that $\hat{\rho}(\mathcal{L},\phi)$ is independent of the choice of the matrix norm $\|\cdot\|$ induced by the vector norm. The following lemma shows that it is also independent of the choice of the matrix P.

LEMMA 2.4. *Suppose $\mathcal{L}(t_0,\phi)\subset\mathcal{H}$ for all $t_0\geq 0$ and $\phi\in\Phi$. Then*

$$\hat{\rho}(\mathcal{L},\phi)=\mathrm{diam}(\mathcal{L},\phi).$$

A proof is given in the appendix.

The Lyapunov exponents are often used to study evolution of the dynamics [5, 6]. Here, we extend the definitions of Lyapunov exponents to general time-varying cases.

DEFINITION 2.5. *For the coupled system* (1.2)*, the Lyapunov exponent of the matrix sequence map $\mathcal{L}$ initiated by $\phi\in\Phi$ in the direction $u\in\mathbb{R}^m$ is defined as*

$$\lambda(\mathcal{L},\phi,u)=\overline{\lim_{t\to\infty}}\frac{1}{t}\sup_{t_0\geq 0}\log\left\|\prod_{k=t_0}^{t+t_0-1}L_k(\varrho^{(k-t_0)}\phi)u\right\|. \tag{2.5}$$

The projection along the synchronization direction e_0 can also define a Lyapunov exponent, called the projection Lyapunov exponent:

$$\hat{\lambda}(\mathcal{L},\phi,v)=\overline{\lim_{t\to\infty}}\frac{1}{t}\sup_{t_0\geq 0}\log\left\|\prod_{k=t_0}^{t+t_0-1}\hat{L}_k(\varrho^{(k-t_0)}\phi)v\right\|, \tag{2.6}$$

where $\hat{L}_k(\varrho^k\phi)$ is the projection of matrix $L_k(\varrho^k\phi)$ as defined in Definition 2.3.

It can be seen that the definition of the generalized Lyapunov exponent above satisfies the basic properties of Lyapunov exponents.[1] For more details about generalized Lyapunov exponents, we refer to [22].

LEMMA 2.6. *Suppose $\mathcal{L}(t_0,\phi)\subset\mathcal{H}$ for all $\phi\in\Phi$ and $t_0\geq 0$. Then*

$$\sup_{v\in\mathbb{R}^{m-1},v\neq 0}\hat{\lambda}(\mathcal{L},\phi,v)=\log\hat{\rho}(\mathcal{L},\phi)=\log\mathrm{diam}(\mathcal{L},\phi).$$

A proof is given in the appendix.

[1] This kind of definition of characteristic exponent is similar to the Bohl exponent used to study uniform stability of time-varying systems in [21].

This lemma implies that the projection joint spectral radius gives the largest Lyapunov exponent in directions transverse to the synchronization direction e_0 of the matrix sequence map $\mathcal{L}$.

When the time dependence arises from being totally driven by some random dynamical system, we can write the generalized matrix sequence map $\mathcal{L}$ as $\mathcal{L}(\phi) = \{L(\varrho^{(t)}\phi)\}_{t\in\mathbb{Z}^+}$ since it is independent of t_0 and is just a map on Φ. As introduced in [23], we have specific definitions for Lyapunov exponents of the time-varying system (2.2) as follows.

For the linear system (2.2), the Lyapunov exponent of the matrix sequence map $\mathcal{L}$ initiated by $\phi \in \Phi$ in the direction $u \in \mathbb{R}^m$ is defined as

$$\lambda(\mathcal{L}, \phi, u) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} L(\varrho^{(k)}\phi)u \right\|. \tag{2.7}$$

If $\mathcal{L}(\phi) \subset \mathcal{H}$ for all $\phi \in \Phi$, then the Lyapunov exponent in the synchronization direction e_0 is

$$\lambda(\mathcal{L}, \phi, e_0) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \sum_{k=0}^{t-1} |c(k)|, \tag{2.8}$$

where $c(k)$ denotes the corresponding common row sum at each time k. The projection along the synchronization direction e_0 can also define a Lyapunov exponent, called the projection Lyapunov exponent:

$$\hat{\lambda}(\mathcal{L}, \phi, v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} \hat{L}_k(\varrho^{(k)}\phi)v \right\|, \tag{2.9}$$

where $\hat{L}(\varrho^k\omega)$ is the (skew) projection of matrix $L(\varrho^k\omega)$. Also, the Hajnal diameter and projection joint spectral radius become

$$\operatorname{diam}(\mathcal{L}, \phi) = \overline{\lim_{t\to\infty}} \left\{ \operatorname{diam}\left(\prod_{k=0}^{t-1} L(\varrho^{(t)}\phi) \right) \right\}^{\frac{1}{t}}, \quad \hat{\rho}(\mathcal{L}, \phi) = \overline{\lim_{t\to\infty}} \left\| \prod_{k=0}^{t-1} \hat{L}(\varrho^{(k)}\phi) \right\|^{\frac{1}{t}}.$$

According to Lemmas 2.4 and 2.6, $\log \operatorname{diam}(\mathcal{L}, \phi) = \log \hat{\rho}(\mathcal{L}, \phi) = \sup_{v\in\mathbb{R}^{m-1}, v\neq 0} \hat{\lambda}$ $(\mathcal{L}, \phi, v)$. Let λ_0 be the Lyapunov exponent along the synchronization direction e_0 and let $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ be the remaining Lyapunov exponents for the initial condition ϕ, counted with multiplicities.

LEMMA 2.7. *Suppose that $\mathcal{L}(\phi) \subset \mathcal{H}$ is time-independent. Let the matrix $D(t) = [D_{ij}(t)]_{i,j=1}^m$ denote the matrix $L(\varrho^{(t)}\phi)$ and let $c(t)$ denote the corresponding common row sum of $D(t)$. If*

1. $\underline{\lim}_{t\to\infty} 1/t \sum_{k=0}^{t-1} \log|c(k)| = \lambda_0$,
2. $\overline{\lim}_{t\to\infty} 1/t \log^+ |D_{ij}(t)| \le 0$ *for all* $i, j = 1, 2, \ldots, m$, *where* $\log^+(z) = \max$ $\{\log z, 0\}$,

then

$$\log \operatorname{diam}(\mathcal{L}, \phi) = \log \hat{\rho}(\mathcal{L}, \phi) = \sup_{i\ge 1} \lambda_i.$$

A proof is given in the appendix.

Using the concept of the Hajnal diameter, we can define (uniform) synchronization of the nonautonomous system (1.2) as follows.

DEFINITION 2.8. *The coupled system* (1.2) *is said to be (uniformly locally completely) synchronized if there exists* $\eta > 0$ *such that for any* $\epsilon > 0$ *there exists* $T > 0$ *such that the inequality*

$$\operatorname{diam}\left([x^1(t), x^2(t), \dots, x^m(t)]^\top\right) \leq \epsilon \tag{2.10}$$

holds for all $t > t_0 + T$, $t_0 \geq 0$, *and* $x^i(t_0)$, $i = 1, 2, \dots, m$, *in the* η *neighborhood of* $s(t_0)$ *of a synchronized state* $s(t)$.

2.2. Stochastic matrix sequences. The above definitions can also be used to deal with stochastic matrix sequences.

DEFINITION 2.9. *A matrix* $G \in \mathbb{R}^{m\times m}$ *is said to be a stochastic matrix if its elements are nonnegative and each row sum is* 1.

We consider here the general time-varying case without the assumption of an underlying random dynamical system and write a stochastic matrix sequence as $\mathcal{G} = \{G(t)\}_{t\in\mathbb{Z}^+}$. The case that the time variation is driven by some dynamical system can be regarded as a special one.

DEFINITION 2.10. *The Hajnal diameter of* $\mathcal{G}$ *is defined as*

$$\operatorname{diam}(\mathcal{G}) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \left(\operatorname{diam} \prod_{k=t_0}^{t_0+t-1} G(k)\right)^{\frac{1}{t}} \tag{2.11}$$

and the projection joint spectral radius for $\mathcal{G}$ *is*

$$\hat{\rho}(\mathcal{G}) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \left\| \prod_{k=t_0}^{t_0+t-1} \hat{G}(k)\right\|^{\frac{1}{t}}, \tag{2.12}$$

where $\hat{G}(t)$ *is the projection of* $G(t)$, *as in Definition* 2.3.

Then, from Lemma 2.4, we have the following.

LEMMA 2.11. $\operatorname{diam}(\mathcal{G}) = \hat{\rho}(\mathcal{G})$.

To estimate the Hajnal diameter of a product of stochastic matrices, we use the concept of scrambling introduced in [20].

DEFINITION 2.12. *A stochastic matrix* $G = [G_{ij}]_{i,j=1}^m \in \mathbb{R}^{m\times m}$ *is said to be scrambling if for any* i, j *there exists an index* k *such that* $G_{ik} \neq 0$ *and* $G_{jk} \neq 0$.

For $g_i = [g_{i,1}, \dots, g_{i,m}] \in \mathbb{R}^m$ and $g_j = [g_{j,1}, \dots, g_{j,m}] \in \mathbb{R}^m$, define

$$g_i \wedge g_j = \left[\min(g_{i,1}, g_{j,1}), \dots, \min(g_{i,m}, g_{j,m})\right].$$

We use the following quantity introduced in [18, 19] to measure scramblingness:

$$\eta(G) = \min_{i,j} \|g_i \wedge g_j\|_1,$$

where $\|\cdot\|_1$ is the norm given by $\|x\|_1 = \sum_{i=1}^m |x_i|$ for $x = [x_1, \dots, x_m] \in \mathbb{R}^m$. It is clear that $0 \leq \eta(G) \leq 1$ and that $\eta(G) > 0$ if and only if G is scrambling. Thus, the well-known Hajnal inequality has the following generalized form.

LEMMA 2.13 (generalized Hajnal inequality [20, Theorem 6]). *For any vector norm in* $\mathbb{R}^m$ *and any two stochastic matrices* G *and* H,

$$\operatorname{diam}(GH) \leq (1 - \eta(G))\operatorname{diam}(H). \tag{2.13}$$

The concepts of projection joint spectral radius and the Hajnal diameter are linked to the ergodicity of stochastic matrix sequences. We can extend the ergodicity for a matrix set [20, 24] to a matrix sequence as follows.

DEFINITION 2.14 (ergodicity [14, Definition 1]). *A stochastic matrix sequence* $\Sigma = \{G(t)\}_{t\in\mathbb{Z}^+}$ *is said to be ergodic if for any* t_0 *and* $\epsilon > 0$ *there exists* $T > 0$ *such that for any* $t > T$ *and some norm* $\|\cdot\|$,

$$\operatorname{diam}\left(\prod_{s=t_0}^{t_0+t-1} G(s)\right) \leq \epsilon. \tag{2.14}$$

Moreover, if for any $\epsilon > 0$ *there exists* $T > 0$ *such that inequality* (2.14) *holds for all* $t \geq T$ *and* $t_0 \geq 0$, $\mathcal{G}$ *is said to be uniformly ergodic.*

A stochastic matrix $G = [G_{ij}]_{i,j=1}^m$ can be associated with a graph $\Gamma = [V, E]$, where $V = \{1, 2, \ldots, m\}$ denotes the vertex set and $E = \{e_{ij}\}$ the edge set, in the sense that there exists an edge from vertex j to i if and only if $G_{ij} > 0$. Let $\Gamma_1 = [V, E_1]$ and $\Gamma = [V, E_2]$ be two simple graphs with the same vertex set. We also define the union $\Gamma_1 \bigcup \Gamma_2 = [V, E_1 \bigcup E_2]$ (merging multiple edges). It can be seen that for two stochastic matrices G_1 and G_2 with the same dimension and positive diagonal elements, the edge set of $\Gamma_1 \bigcup \Gamma_2$ is contained in that of the corresponding graph of the product matrix G_1G_2. In this way, we can define the union of the graph sequence $\{\Gamma(t)\}_{t\in\mathbb{Z}^+}$ across the time interval $[t_1, t_2]$ by $\bigcup_{k=t_1}^{t_2} \Gamma(k) = [V, \bigcup_{k=t_1}^{t_2} E(k)]$. The following concepts for graphs can be found, e.g., in [25].

DEFINITION 2.15. *A graph* Γ *is said to have a spanning tree if there exists a vertex, called the root, such that for each other vertex* j *there exists at least one directed path from the root to vertex* j.

It follows that $\{\Gamma(t)\}_{t\in\mathbb{Z}^+}$ has a spanning tree across the time interval $[t_1, t_2]$ if the union of $\{\Gamma(t)\}_{t\in\mathbb{Z}^+}$ across $[t_1, t_2]$ has a spanning tree. This is equivalent to the existence of a vertex from which all other vertices can be accessible across $[t_1, t_2]$.

DEFINITION 2.16. *A graph* Γ *is said to be scrambling if for any different vertices* i *and* j *there exists a vertex* k *such that there exist edges from* k *to* i *and from* k *to* j.

It follows that a stochastic matrix G is scrambling if and only if the corresponding graph Γ is scrambling.

LEMMA 2.17 (see [24, Lemma 4]). *Let* $G(1), G(2), \ldots, G(m-1)$ *be stochastic matrices with positive diagonal elements, where each of the corresponding graphs* $\Gamma(1)$, $\Gamma(2), \ldots, \Gamma(m-1)$ *have spanning trees. Then* $\prod_{k=1}^{m-1} G(k)$ *is scrambling.*

Suppose now that the stochastic matrix sequence $\mathcal{G}$ is driven by some metric dynamical system $\mathcal{Y} = \{\Omega, \mathcal{F}, P, \theta^{(t)}\}$. We write $\mathcal{G}$ as $\{G(t) = G(\theta^{(t)}\omega)\}_{t\in\mathbb{Z}^+}$, where $\omega \in \Omega$. Then, as stated in section 2.1, we can define the Lyapunov exponents.

DEFINITION 2.18. *The Lyapunov exponent of the stochastic matrix sequence* $\mathcal{G}$ *is defined as*

$$\sigma(\mathcal{G}, \omega, u) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} G(\theta^t \omega) u \right\|.$$

The projection Lyapunov exponents is defined as

$$\hat{\sigma}(\mathcal{G}, \omega, u) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} \hat{G}(\theta^t \omega) u \right\|,$$

where $\hat{G}(\cdot)$ *is the projection of* $G(\cdot)$ *as defined in Definition* 2.3.

For a given $\omega \in \Omega$, one can see that $\operatorname{diam}(\mathcal{G})$ and $\hat{\rho}(\mathcal{G})$ both equal the largest Lyapunov exponent of $\mathcal{G}$ in directions transverse to the synchronization direction under several mild conditions.

In closing this section, we list some notation to be used in the remainder of the paper. The matrix $\hat{L}$ denotes the (skew) projection of the matrix L along the vector e introduced in Definition 2.3, and $\hat{\mathcal{L}}$ is the (skew) projection of the matrix sequence map $\mathcal{L}$ along e. For $x = (x^1, \dots, x^m)^\top \in \mathbb{R}^m$, the average $\frac{1}{m}\sum_{i=1}^m x^i$ of x is denoted by $\bar{x}$. The notation $\|\cdot\|$ denotes some vector norm in the linear space $\mathbb{R}^m$, and also the matrix norm in $\mathbb{R}^{m\times m}$ induced by this vector norm. $f^{(t)}(s_0)$ denotes the t-iteration of the map f with initial condition s_0. We let $x(t, t_0, x_0)$ be the solution of the coupled system (1.2) with initial condition $x(t_0) = x_0$, which we sometimes abbreviate as $x(t)$.

3. Generalized synchronization analysis. For the variational system (1.12), similar to subsection 2.1, we denote by $\mathcal{D}$ the Jacobian sequence map in the generalized sense; i.e., $\mathcal{D}$ is a map from $\mathbb{Z}^+ \times \mathbb{R}$ to $2^{\mathbb{R}^{m\times m}}$: $\mathcal{D}(t_0, s_0) = \{D_{t+t_0}(f^{(t)}(s_0))\}_{t\in\mathbb{Z}^+} \subset \mathcal{H}$ for all $t_0 \in \mathbb{Z}^+$ and $s_0 \in A$. Furthermore, letting

$$B(t, t_0) = \prod_{k=t_0}^{t+t_0-1} D_k(f^{(k-t_0)}(s_0)),$$

we can rewrite the variational system (1.12) as follows:

$$\delta x(t+t_0) = D_{t+t_0-1}(f^{(t-1)}(s_0))\delta x(t+t_0-1) = B(t, t_0)\delta x(t_0). \tag{3.1}$$

From Definitions 2.2 and 2.3, we have

$$\operatorname{diam}(\mathcal{D}, s_0) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \left\{ \operatorname{diam}\left(\prod_{k=t_0}^{t_0+t-1} D_k(f^{(k-t_0)}(s_0)) \right) \right\}^{\frac{1}{t}},$$

$$\hat{\rho}(\mathcal{D}, s_0) = \overline{\lim_{t\to\infty}} \sup_{t_0\geq 0} \left\| \prod_{k=t_0}^{t_0+t-1} \hat{D}_k(f^{(k-t_0)}(s_0)) \right\|^{\frac{1}{t}}.$$

We will also refer to the following hypothesis.

(H_3)

$$\sup_{s_0\in A} \operatorname{diam}(\mathcal{D}, s_0) < 1. \tag{3.2}$$

THEOREM 3.1. *If hypotheses* (H_1)–(H_3) *hold, then the compact set* $A^m \bigcap \mathcal{S}$ *is a uniformly asymptotically stable attractor of the coupled system* (1.2) *in* $\mathbb{R}^m$; *i.e., the coupled system* (1.2) *is uniformly locally completely synchronized.*

Proof. Let

$$\operatorname{diam}(\mathcal{D}, t_0, t, s_0) = \operatorname{diam}\left(\prod_{k=t_0}^{t_0+t-1} D_k\big(f^{(k-t_0)}(s_0)\big) \right),$$

$$\operatorname{diam}(\mathcal{D}, t, s_0) = \sup_{t_0\geq 0} \left\{ \operatorname{diam}\left(\prod_{k=t_0}^{t_0+t-1} D_k(f^{(k-t_0)}(s_0)) \right) \right\}.$$

According to (H_3), letting $1 > d > \sup_{s_0\in A} \operatorname{diam}(\mathcal{D}, s_0)$ and n_0 satisfy $d^{n_0} < \frac{1}{3}$, for any $s_0 \in A$, there exists $n(s_0) \geq n_0$ such that $\operatorname{diam}(\mathcal{D}, t, s_0) < d$ holds for all $t \geq n(s_0)$. By equicontinuity (H_1) and compactness (H_2), there must exist a finite integer set $\mathcal{V} = \{n_1, n_2, \ldots, n_v\}$ satisfying $n_i \geq n_0$ for all $i = 1, 2, \ldots, v$ and a neighborhood U of A such that for any $s_0 \in U$ there exists $n_j \in \mathcal{V}$ such that $\mathrm{diam}\big(\prod_{k=t_0}^{t_0+n_j-1} D_k(f^{(k-t_0)}(s_0))\big) < d^{n_j} < \frac{1}{3}$ holds for all $t_0 \geq 0$.

By hypothesis (H_2), there exists a compact neighborhood W of A such that $U \supset W \supset A$, $f(W) \subset W$, and $\bigcap_{n\geq 0} f^{(n)}(W) = A$ [26]. Let

$$a = \min_{n\in\mathcal{V}} d_H(f^{(n)}(W), W) > 0,$$

where $d_H(\cdot,\cdot)$ denotes the Hausdorff metric in $\mathbb{R}$. Then define a compact set

$$W_\alpha = \left\{ x = (x^1, \ldots, x^m) \in \mathbb{R}^m : \max_{1\leq i\leq m} |x^i - \bar{x}| \leq \alpha \text{ and } \bar{x} \in W \right\}.$$

By the mean value theorem, we have

$$f_k^i(x^1(k), \ldots, x^m(k)) - f(s(k), \ldots, s(k)) = \sum_{j=1}^m \frac{\partial f_k^i}{\partial x^j}(\xi_k^{ij}),$$

where ξ_k^{ij} belongs to the closed interval induced by the two ends $x^i(k)$ and $s(k)$. Denote by $D_k(\xi_k)$ the matrix $[\partial f_k^i(\xi_k^{ij})/\partial x^j]_{i,j=1}^m$.

Let $\alpha > 0$ be sufficiently small so that for each $x_0 \in W_\alpha$ with $s(t_0) = \bar{x}_0$ and $x(t_0) = x_0$, there exists $t_1 \in \mathcal{V}$ such that

$$|x^i(t_1, t_0, x_0) - f^{(t_1-t_0)}(\bar{x}_0)| \leq \frac{a}{2},$$
$$\mathrm{diam}\left(\prod_{k=t_0}^{t_0+t_1-1} D_k(\xi_k) \right) < \frac{1}{2}$$

hold for all $t_0 \geq 0$. Then, for any $x_0 \in W_\alpha$, $\bar{x}_0 \in W$, we have

$$\delta x(t_1 + t_0) = \prod_{k=t_0}^{t_1+t_0-1} D_k(\xi(k))\delta x_0 = \tilde{B}(t_1, t_0)\delta x_0,$$

where $\tilde{B}(t_1, t_0) = \prod_{k=t_0}^{t_1+t_0-1} D_k(\xi(k))$. Then

$$\begin{aligned} |\delta x^i(t_1 + t_0) - \delta x^j(t_1 + t_0)| &\leq \sum_{k=1}^m |\tilde{B}_{ik}(t_1, t_0) - \tilde{B}_{jk}(t_1, t_0)||\delta x_0^j| \\ &\leq \mathrm{diam}(\tilde{B}(t_1, t_0)) \max_{1\leq i\leq m} |x_0^i - \bar{x}_0|. \end{aligned}$$

Thus, we conclude that

$$\max_{1\leq i,j\leq m} |x^i(t_1 + t_0) - x^j(t_1 + t_0)| \leq \frac{1}{2} \max_{1\leq i,j\leq m} |x_0^i - x_0^j|.$$

By the definition of W_α, we see that $x(t_1 + t_0) \in W_{\alpha/2}$. With initial time $t_0 + t_1$, we can continue this phase and afterwards obtain

$$\lim_{t\to\infty} |x^i(t) - x^j(t)| = 0, \ i, j = 1, 2, \ldots, m,$$

uniformly with respect to $t_0 \in \mathbb{Z}^+$ and $x_0 \in W_\alpha$. Therefore, the coupled system (1.2) is uniformly synchronized. Furthermore, we obtain that $A^m \bigcap \mathcal{S}$ is a uniformly asymptotically stable attractor for the coupled system (1.2), and the convergence rate can be estimated by $O(\{\sup_{s_0\in A} \mathrm{diam}(\mathcal{D}, s_0)\}^t)$ since d is chosen arbitrarily greater than $\sup_{s_0\in A} \mathrm{diam}(\mathcal{D}, s_0)$. The theorem is proved. ∎

Remark 1. The idea of the above proof comes from that of Theorem 2.12 in [1], with a modification for the time-varying case. In Theorem 2.12 in [1], the authors used normal Lyapunov exponents to prove asymptotic stability of the original autonomous system for the case when it is asymptotically stable in an invariant manifold. In this paper, we directly use the Hajnal diameter of the left product of the infinite Jacobian matrix sequence map to measure the transverse differences of the collections of spatial states. Furthermore, we consider a nonautonomous system here due to time-varying couplings.

Following Lemma 2.4 gives us the following.

COROLLARY 3.2. *If* $\sup_{s_0\in A} \hat\rho(\mathcal{D}, s_0) < 1$, *then the coupled system* (1.2) *is uniformly synchronized.*

Consider the special case that the coupled system (1.9) is an RDS on an MDS $\mathcal{Y} = \{\Omega, \mathcal{F}, P, \theta^{(t)}\}$. We can write this coupled system (1.9) as a product dynamical system $\{A\times\Omega, \mathbf{F}, \mathbf{P}, \Theta^{(t)}\}$, where $\mathbf{F}$ is the product σ-algebra on $A\times\Omega$, $\mathbf{P}$ denotes the probability measure, and $\Theta^{(t)}(s_0, \omega) = (\theta^{(t)}\omega, f^{(t)}(s_0))$. Let $D(f^{(t)}(s_0), \theta^{(t)}\omega)$ denote the Jacobian matrix at time t. By Definition 2.5, the Lyapunov exponents for the coupled system (1.9) can be written as follows:

$$\lambda(u, s_0, \omega) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} D(f^{(k)}(s_0), \theta^{(k)}\omega) u \right\|.$$

It can be seen that the Lyapunov exponent along the diagonal synchronization direction e_0 is

$$\lambda(e_0, s_0, \omega) = \overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{k=0}^{t-1} \log |c(k)|,$$

where $c(k)$ is the common row sum of $D(f^{(k)}(s_0), \theta^{(k)}\omega)$. Let $\lambda_0 = \lambda(e_0, s_0, \omega)$, $\lambda_1, \ldots, \lambda_{m-1}$ be the Lyapunov exponents (counting multiplicity) of the dynamical system $\mathcal{L}$ with the initial condition (s_0, ω). From Lemma 2.7, we conclude that $\sup_{i\geq 1} \lambda_i = \log \hat\rho(F, s_0, \omega) = \log \mathrm{diam}(F, s_0, \omega)$. If the probability $\mathbf{P}$ is ergodic, then the Lyapunov exponents exist for almost all $s_0 \in A$ and $\omega \in \Omega$, and furthermore they are independent of (s_0, ω).

COROLLARY 3.3. *Suppose that hypotheses* (H_1)–(H_2) *and the assumptions in Lemma* 2.7 *hold. Suppose further that* $A\times\Omega$ *is compact in the weak topology defined in this RDS, the semiflow* $\Theta^{(t)}$ *is continuous, the Jacobian matrix* $D(\cdot,\cdot)$ *is nonsingular and continuous on* $A\times\Omega$, *and*

$$\sup_{\mathbf{P}\in \mathrm{Erg}_\Theta(\mathrm{A}\times\Omega)} \sup_{i\geq 1} \lambda_i < 0,$$

where $\mathrm{Erg}_\Theta(\mathrm{A}\times\Omega)$ *denotes the ergodic probability measure set supported in* $\{A\times\Omega, \mathbf{F}, \Theta^{(t)}\}$. *Then the coupled system* (1.9) *is uniformly locally completely synchronized.*

Proof. By Theorem 2.8 in [1], we have

$$\sup_{\mathbf{P}\in Erg_\Theta(A\times\Omega)} \lambda_{\max}(\hat{\mathcal{D}}, \mathbf{P}) = \sup_{\|u\|=1, (s_0,\omega)\in A\times\Omega} \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} \hat D(f^{(k)}(s_0), \theta^{(k)}\omega) u \right\|,$$

where $\hat{\mathcal{D}}$ is the projection of the intrinsic matrix sequence map $\mathcal{D}$ and $\lambda_{\max}(\hat{\mathcal{D}}, \mathbf{P})$ denotes the largest Lyapunov exponent of $\hat{\mathcal{D}}$ according to the ergodic probability $\mathbf{P}$ (the value for almost all (s_0, ω) according to $\mathbf{P}$). From Lemmas 2.4, 2.6, and 2.7, it follows that

$$\sup_{\mathbf{P}\in \mathrm{Erg}_{\Theta}(\mathrm{A}\times\Omega)} \sup_{i\geq 1} \lambda_i = \sup_{\mathbf{P}\in \mathrm{Erg}_{\Theta}(\mathrm{A}\times\Omega)} \lambda_{\max}(\hat{\mathcal{D}}, \mathbf{P}) = \sup_{(s_0,\omega)\in A\times\Omega} \lambda_{\max}(\hat{\mathcal{D}}, s_0, \omega)$$
$$= \sup_{(s_0,\omega)\in A\times\Omega} \log \hat{\rho}(\mathcal{D}, s_0, \omega) = \sup_{(s_0,\omega)\in A\times\Omega} \log \mathrm{diam}(\mathcal{D}, s_0, \omega).$$

The corollary is proved as a direct consequence from Theorem 3.1. ∎

Remark 2. If λ_0 is the largest Lyapunov exponent, then $V = \{u : \ \lambda(u) < \lambda_0\}$ constructs a subspace of $\mathbb{R}^m$ which is transverse to the synchronization direction e_0. Corollary 3.3 implies that if all Lyapunov exponents in the transverse directions are negative, then the coupled system (1.2) is synchronized. Otherwise, if λ_0 is not the largest Lyapunov exponent, then $\sup_{i\geq 1} \lambda_i < 0$ implies that the largest exponent is negative, which means that the synchronized solution $s(t)$ is itself asymptotically stable through the evolution (1.9).

Remark 3. From Lemma 2.7, it can also be seen that when computing $\rho(\mathcal{D})$, it is sufficient to compute the largest Lyapunov exponent of $\hat{\mathcal{D}}$. In [1], the authors proved for an autonomous dynamical system that if all Lyapunov exponents of the normal directions, namely, the Lyapunov exponents for $\hat{\mathcal{D}}$, are negative, then the attractor in the invariant submanifold is an attractor in $\mathbb{R}^m$ (or a more general manifold). In this paper, we extend the proof of Theorem 2.12 in [1] to the general time-varying coupled system (1.2) by discussing the relation between the Hajnal diameter and transverse Lyapunov exponents. In the following sections, we continue the synchronization analysis for nonautonomous dynamical systems.

4. Synchronization analysis of CML with time-varying topologies. Consider the following coupled system with time-varying topologies:

$$x^i(t+1) = \sum_{j=1}^{m} G_{ij}(t) f(x^j(t)), \quad i = 1, 2, \ldots, m, \ t \in \mathbb{Z}^+, \tag{4.1}$$

where $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is C^1 continuous and $G(t) = [G_{ij}(t)]_{i,j=1}^m$ is a stochastic matrix. In matrix form,

$$x(t+1) = G(t)F(x(t)). \tag{4.2}$$

Since the coupling matrix $G(t)$ is a stochastic matrix, the diagonal synchronization manifold is invariant and we have the uncoupled (or synchronized) state as

$$s(t+1) = f(s(t)). \tag{4.3}$$

We suppose that for the synchronized state (4.3), there exists an asymptotically stable attractor A with the (maximum) Lyapunov exponent

$$\mu = \sup_{s_0\in A} \overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{k=0}^{t-1} \log |f'(s(k))|.$$

System (4.1) is a special form of (1.2) satisfying the equicontinuous condition (H_1). Linearizing system (4.1) about the synchronized state yields the variational equation

$$\delta x^i(t+1) = \sum_{j=1}^{m} G_{ij}(t) f'(s(t)) \delta x^i(t), \ i = 1, 2, \ldots, m,$$

and

$$\operatorname{diam}\left(\prod_{k=t_0}^{t_0+t-1} G(k) f'(f^{(k-t_0)}(s_0))\right) = \operatorname{diam}\left(\prod_{k=t_0}^{t_0+t-1} G(k)\right)\left|\prod_{l=0}^{t} f'(f^{(l)}(s_0))\right|.$$

Denote the stochastic matrix sequence $\{G(t)\}_{t\in\mathbb{Z}^+}$ by $\mathcal{G}$. Thus, the Hajnal diameter of the variational system is $\operatorname{diam}(\mathcal{G})e^{\mu}$. Using Theorem 3.1, we have the following result.

THEOREM 4.1. *Suppose that the uncoupled system $s(t+1) = f(s(t))$ satisfies hypothesis* (H_2) *with Lyapunov exponent μ. Let $\mathcal{G} = \{G(t)\}_{t\in\mathbb{Z}^+}$. If*

$$\operatorname{diam}(\mathcal{G})e^{\mu} < 1, \tag{4.4}$$

then the coupled system (4.1) *is synchronized.*

From Theorem 4.1, one can see that the quantity $\operatorname{diam}(\mathcal{G})$ as well as other equivalent quantities such as the projection joint spectral radius and the Lyapunov exponent, can be used to measure the synchronizability of the time-varying coupling, i.e., the coupling stochastic matrix sequence $\mathcal{G}$. A smaller value of $\operatorname{diam}(\mathcal{G})$ implies a better synchronizability of the time-varying coupling topology. If the uncoupled system (4.3) is chaotic, i.e., $\mu > 0$, then the necessary condition for synchronization condition (4.4) is $\operatorname{diam}(\mathcal{G}) < 1$. So, it is important to investigate under what conditions $\operatorname{diam}(\mathcal{G}) < 1$ holds.

Suppose that the stochastic matrix set $\mathcal{M}$ satisfies the following hypotheses.

(H_4) *$\mathcal{M}$ is compact and there exists $r > 0$ such that for any $G = [G_{ij}]_{i,j=1}^m \in \mathcal{M}$, $G_{ij} > 0$ implies $G_{ij} \geq r$ and all diagonal elements $G_{ii} > r$, $i = 1, 2, \ldots, m$.*

We denote the graph sequence corresponding to the stochastic matrix sequence $\mathcal{G}$ by $\mathbf{\Gamma} = \{\Gamma(t)\}_{t\in\mathbb{Z}^+}$. Then we have the following result.

THEOREM 4.2. *Suppose that the stochastic matrix sequence $\mathcal{G} \subset \mathcal{M}$ satisfies hypothesis* (H_4). *Then the following statements are equivalent:*

1. $\operatorname{diam}(\mathcal{G}) < 1$;
2. *there exists $T > 0$ such that for any t_0 the graph $\bigcup_{k=t_0}^{t_0+T}\Gamma(k)$ has a spanning tree*;
3. *the stochastic matrix sequence $\mathcal{G}$ is uniformly ergodic.*

Proof. We first show that (3) ⇒ (2) by reduction to absurdity. Let $B(t_0, t) = \prod_{k=t_0}^{t_0+t-1} G(k)$. Since $\mathcal{G}$ is uniformly ergodic, there must exist $T > 0$ such that $\operatorname{diam}(B(t_0, T)) < 1/2$ holds for any $t_0 \geq 0$. So, $v = \prod_{k=t_0}^{t_0+T-1} G(k)u$ satisfies

$$\max_{1\leq i,j\leq m} |v_i - v_j| \leq \operatorname{diam}(B(t_0, T))\|u\|_\infty \leq \frac{1}{2}\|u\|_\infty. \tag{4.5}$$

If the second condition does not hold, then there exists t_T such that the union $\bigcup_{k=t_T}^{t_T+T-1}\Gamma(k)$ does not have a spanning tree. That is, there exist two vertices v_1 and v_2 such that for any vertex z there is either no directed path from z to v_1 or no directed path from z to v_2. Let U_1 (U_2) be the vertex set which can reach v_1 (v_2, respectively) across $[t_T, t_T + T - 1]$. This implies that U_1 and U_2 are disjoint across $[t_T, t_T + T - 1]$ and no edge starts outside of U_1 (U_2) and ends in U_1 (U_2). Furthermore, considering the Frobenius form of $G(t)$, one can see that the elements in the corresponding rows of U_1 (U_2) with columns associated with the complementary set of U_1 (U_2) are all zeros. Let

$$u_i = \begin{cases} 1, & i \in U_1, \\ 0, & i \in U_2, \\ \text{any value in } (0,1), & \text{otherwise.} \end{cases}$$

We have

$$v_i = \begin{cases} 1, & i \in U_1, \\ 0, & i \in U_2, \\ \in [0,1], & \text{otherwise.} \end{cases}$$

This implies that $\max_{1\le i,j\le m}|v_i - v_j| \ge 1 = \|u\|_\infty$, which contradicts (4.5). Therefore, (3) $\Rightarrow$ (2) can be concluded.

We next show (2) $\Rightarrow$ (1). Applying Lemma 2.17, there exists $T > 0$ such that $\prod_{k=t_0}^{t_0+T-1} G(k)$ is scrambling for any t_0. There exists $\delta > 0$ such that $\eta(B(T,t_0)) > \delta > 0$ for all $t_0 \ge 0$ because of the compactness of the set $\mathcal{M}$. So,

$$\begin{aligned} \operatorname{diam}(B(t,t_0)) &= \operatorname{diam}\left\{ B\left(\operatorname{mod}(t,T), t_0 + \left[\frac{t}{T}\right] T\right) \prod_{k=1}^{[\frac{t}{T}]} B(T, t_0 + (k-1)T) \right\} \\ &\le \operatorname{diam}\left\{ \prod_{k=1}^{[\frac{t}{T}]} B(t_0 + kT - 1, t_0 + (k-1)T) \right\} \\ &\le 2(1-\delta)^{[\frac{t}{T}]} \end{aligned} \tag{4.6}$$

holds for any $t_0 \ge 0$. Here, $[t/T]$ denotes the largest integer less than t/T and $\operatorname{mod}(t,T)$ denotes the modulus of the division $t \div T$. Thus,

$$\operatorname{diam}(\mathcal{G}) \le (1-\delta)^{\frac{1}{T}} < 1.$$

This proves (2) $\Rightarrow$ (1). Since (1) $\Rightarrow$ (3) is clear, the theorem is proved. ∎

Remark 4. According to Lemma 2.17, it can be seen that the union of graphs across any time interval of length T has a spanning tree if and only if a union of graphs across any time interval of length $(m-1)T$ is scrambling.

Moreover, from [27], we conclude more results on the ergodicity of stochastic matrix sequences as follows.

PROPOSITION 4.3. *The implication* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *holds for the following statements:*

1. $\operatorname{diam}(\mathcal{G}) < 1$*;*
2. $\mathcal{G}$ *is ergodic;*
3. *for any* $t_0 \ge 0$*, the union* $\bigcup_{k\ge t_0} \Gamma(k)$ *has a spanning tree.*

Remark 5. It should be pointed out that the implications in Proposition 4.3 cannot be reversed. Counterexamples can be found in [14]. However, in [14], it is also proved under certain conditions that if the stochastic matrices have the property that $G_{ij} > 0$ if and only if $G_{ji} > 0$, then statement 2 is equivalent to statement 3.

Assembling Theorem 4.2, Proposition 4.3, and the results in [14], it can be shown that, for $\mathcal{G} \subset \mathcal{M}$, the implications

$$A_1 \Leftrightarrow A_2 \Leftrightarrow A_3 \Rightarrow A_4 \Rightarrow A_5$$

hold regarding the following statements:

- $\mathcal{A}_1$: $\operatorname{diam}(\mathcal{G}) < 1$.
- $\mathcal{A}_2$: there exists $T > 0$ such that the union across any T-length time interval $[t_0.t_0+T]$: $\bigcup_{k=t_0}^{t_0+T} \Gamma(k)$ has a spanning tree.
- $\mathcal{A}_3$: $\mathcal{G}$ is uniformly ergodic.
- $\mathcal{A}_4$: $\mathcal{G}$ is ergodic.

- $\mathcal{A}_5$: for any t_0, the union across $[t_0, \infty)$: $\bigcup_{k \geq t_0} \Gamma(k)$ has a spanning tree.

In the following, we present some special classes of examples of CML with time-varying couplings. These classes were widely used to describe discrete-time networks and were studied in some recent papers [5, 6, 20, 27]. The synchronization criterion for these classes can be verified by numerical methods. Thus, the synchronizability $\mathrm{diam}(\mathcal{G})$ of the time-varying couplings can also be computed numerically.

4.1. Static topology. If $G(t)$ is a static matrix, i.e., $G(t) = G$ for all $t \in \mathbb{Z}^+$, then we can write the coupled system (4.1) as

$$x(t+1) = GF(x(t)). \tag{4.7}$$

PROPOSITION 4.4. *Let* $1 = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ *be the eigenvalues of* G *ordered by* $1 \geq |\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_{m-1}|$. *If* $|\sigma_1| e^{\mu} < 1$, *then the coupled system* (4.7) *is synchronized.*

Proof. Let $v_0 = e_0$ and choose column vectors $v_1, v_2, \ldots, v_{m-1}$ in $\mathbb{R}^m$ such that $v_0, v_1, \ldots, v_{m-1}$ is an orthonormal basis for $\mathbb{R}^m$. Let $A = [v_0, v_1, \ldots, v_{m-1}]$. Then

$$A^{-1}GA = \begin{bmatrix} 1 & \alpha \\ 0 & \hat{G} \end{bmatrix},$$

where the eigenvalues of $\hat{G}$ are $\sigma_1, \ldots, \sigma_{m-1}$. By the Householder theorem (see Theorem 4.2.1 in [28]), for any $\epsilon > 0$, there must exist a norm in $\mathbb{R}^m$ such that with its induced matrix norm,

$$|\sigma_1| \leq \|\hat{G}\| \leq |\sigma_1| + \epsilon.$$

Since ϵ is arbitrary, for the static stochastic matrix sequence $\mathcal{G}_0 = \{G, G, \ldots,\}$, it can be concluded that $\hat{\rho}(\mathcal{G}_0) = |\sigma_1|$. Using Theorem 4.1, the conclusion follows. Moreover, it can also be obtained that the convergence rate is $O((|\sigma_1| e^{\mu})^t)$. ∎

Remark 6. Similar results have been obtained by several papers concerning synchronization of CML with static connections (see [5, 6, 8, 29]). Here, we have proved this result in a different way as a consequence of our main result.

4.2. Finite topology set. Let $\mathcal{Q}$ be a compact stochastic matrix set satisfying (H_4). Consider the following inclusions:

$$x(t+1) \in \mathcal{Q}F(x(t)), \tag{4.8}$$

i.e.,

$$x(t+1) = G(t)F(x(t)), \tag{4.9}$$

$$G(t) \in \mathcal{Q}. \tag{4.10}$$

Then the synchronization of the coupled system (4.8) can be formulated as follows.

DEFINITION 4.5. *The coupled inclusion system* (4.8) *is said to be synchronized if, for any stochastic matrix sequence* $\mathcal{G} \subset \mathcal{Q}$, *the coupled system* (4.9) *is synchronized.*

In [20], the authors defined the Hajnal diameter and projection joint spectral radius for a compact stochastic matrix set.

DEFINITION 4.6. *For the stochastic matrix set* $\mathcal{Q}$, *the Hajnal diameter is given by*

$$\mathrm{diam}(\mathcal{Q}) = \overline{\lim_{t \to \infty}} \sup_{G(k) \in \mathcal{Q}} \left\{ \mathrm{diam}\left(\prod_{k=0}^{t-1} G(k) \right) \right\}^{\frac{1}{t}},$$

and the projection joint spectral radius is

$$\hat{\rho}(\mathcal{Q}) = \overline{\lim_{t\to\infty}} \left\{ \sup_{G(k)\in\mathcal{Q}} \left\| \prod_{k=0}^{t-1} \hat{G}(k) \right\| \right\}^{\frac{1}{t}}.$$

The following result is from [20].

LEMMA 4.7. *Suppose $\mathcal{Q}$ is a compact set of stochastic matrices. Then*

$$\operatorname{diam}(\mathcal{Q}) = \hat{\rho}(\mathcal{Q}).$$

Using Theorem 4.1, we have the following.

THEOREM 4.8. *If* $\operatorname{diam}(\mathcal{Q})e^{\mu} < 1$, *then the coupled system* (4.8) *is synchronized.*

Moreover, we conclude that the synchronization is uniform with respect to $t_0 \in \mathbb{Z}^+$ and stochastic matrix sequences $\mathcal{G} \subset \mathcal{Q}$. Furthermore, we have the following result on synchronizability of the stochastic matrix set $\mathcal{Q}$.

PROPOSITION 4.9. *Let $\mathcal{Q}$ be a compact set of stochastic matrices satisfying hypothesis* (H_4). *Then the following statements are equivalent:*

- $\mathcal{B}_1$: $\operatorname{diam}(\mathcal{Q}) < 1$.
- $\mathcal{B}_2$: *for any stochastic matrix sequence $\mathcal{G} \subset \mathcal{Q}$, $\mathcal{G}$ is ergodic.*
- $\mathcal{B}_3$: *each corresponding graph of a stochastic matrix $G \in \mathcal{Q}$ has a spanning tree.*

Proof. The implication $\mathcal{B}_1 \Rightarrow \mathcal{B}_2 \Rightarrow \mathcal{B}_3$ is clear by Proposition 4.3. And $\mathcal{B}_3 \Rightarrow \mathcal{B}_1$ can be obtained by the proof of Theorem 4.2 since $\mathcal{Q}$ is a finite set of stochastic matrices satisfying hypothesis (H_4). ∎

Remark 7. By the methods introduced in [30, 31, 32], $\hat{\rho}(\mathcal{Q})$ can be computed to arbitrary precision for a finite set $\mathcal{Q}$ despite a large computational complexity.

4.3. Multiplicative ergodic topology sequence. Consider the stochastic matrix sequence $\mathcal{G} = \{G(t)\}_{t\in\mathbb{Z}^+}$ driven by some dynamical system $\mathcal{Y} = \{\Omega, \mathcal{F}, P, \theta^{(t)}\}$, i.e., $\mathcal{G} = \{G(\theta^{(t)}\omega)\}$ for some continuous map $G(\cdot)$. Recall the Lyapunov exponent for $\mathcal{G}$:

$$\sigma(v,\omega) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left\| \prod_{k=0}^{t-1} G(\theta^{(k)}\omega)v \right\|.$$

It is clear that $\sigma(e_0, \omega) = 0$ for all ω and $\sigma(v,\omega) \le 0$ for all ω and $v \in \mathbb{R}^m$. So, the linear subspace

$$L_\omega = \{v, \sigma(v,\omega) < 0\}$$

denotes the directions transverse to the synchronization manifold. If P is an ergodic measure for the MDS $\mathcal{Y}$, then $\sigma(u,\omega)$ and $L(\omega)$ are the same for almost all ω with respect to P [33]. Then we can let σ_1 be the largest Lyapunov exponent of $\mathcal{G}$ transverse to the synchronization direction e_0. By Theorem 4.1 and Corollary 3.3, we have the following.

THEOREM 4.10. *Suppose that $\theta^{(t)}$ is a continuous semiflow*, $G(\cdot)$ *is continuous on all $\omega \in \Omega$ and nonsingular*, *and Ω is compact. If*

$$\sup_{\mathrm{Erf}_\theta(\Omega)} \sigma_1 + \mu < 0,$$

then the coupled system (4.1) *is synchronized.*

Remark 8. There are many papers discussing the computation of multiplicative Lyapunov exponents; for example, see [34, 35]. In particular, [36] discussed the Lyapunov exponents for the product of infinite matrices. By Lemma 2.7, we can compute the largest projection Lyapunov exponent which equals σ_1. We will illustrate this in the following section.

5. Numerical illustrations. In this section, we will numerically illustrate the theoretical results on synchronization of CML with time-varying couplings. In these examples, the coupling matrices are driven by random dynamical systems which can be regarded as stochastic processes. Then the projection Lyapunov exponents are computed numerically by the time series of coupling matrices. In this way, we can verify the synchronization criterion and analyze synchronizability numerically. Consider the following coupled map network with time-varying topology:

$$x^i(t+1) = \frac{1}{\sum\limits_{k=1}^{m} A_{ik}(t)} \sum_{j=1}^{m} A_{ij}(t) f(x^j(t)),\ \ i = 1, 2, \ldots, m, \tag{5.1}$$

where $x^i(t) \in \mathbb{R}$ and $f(s) = \alpha s(1-s)$ is the logistic map with $\alpha = 3.9$, which implies that the Lyapunov exponent of f is $\mu \approx 0.5$. The stochastic coupling matrix at time t is

$$G(t) = [G_{ij}(t)]_{i,j=1}^{m} = \left[\frac{A_{ij}(t)}{\sum\limits_{j=1}^{m} A_{ij}(t)}\right]_{i,j=1}^{m}.$$

5.1. Blinking scale-free networks. The blinking scale-free network is a model initiated by a scale-free network and evolves with malfunction and recovery. At time $t = 0$, the initial graph $\Gamma(0)$ is a scale-free network introduced in [37]. At each time $t \geq 1$, every vertex i malfunctions with probability $p \ll 1$. If vertex i malfunctions, all edges linked to it disappear. In addition, a malfunctioned vertex recovers after a time interval T and then causes the reestablishment of all edges linked to it in the initial graph $\Gamma(0)$. The coupling $A_{ij}(t) = A_{ji}(t) = 1$ if vertex j is connected to i at time t; otherwise, $A_{ij}(t) = A_{ji}(t) = 0$ and $A_{ii}(t) = 1$ for all $i, j = 1, 2, \ldots, m$.

In Figure 1, we show the convergence of the second Lyapunov exponent σ_1 during the topology evolution with different malfunction probability p. We measure synchronization by the variance $K = 1/(m-1)\langle\sum_{i=1}^{m}(x^i(t) - \bar{x}(t))^2\rangle$, where $\langle\cdot\rangle$ denotes the time average, and we denote $W = \sigma_1 + \mu$. We pick the evolution time length to be 1000 and choose initial conditions randomly from the interval $(0, 1)$. In Figure 2, we show the variation of K and W with respect to the malfunction probability p. It can be seen that the region where W is negative coincides with the region of synchronization, i.e., where K is near zero.

5.2. Blurring directed graph process. A blurring directed graph process is one where each edge weight is a modified Wiener process. In details, the graph process is started with a directed weighted graph $\Gamma(0)$ of which for each vertex pair (i, j), one of two edges $A_{ij}(0)$ and $A_{ji}(0)$ is a random variable uniformly distributed between 1 and 2, and the other is zero with equal probability, for all $i \neq j$; $A_{ii}(0) = 0$ for all $i = 1, 2, \ldots, m$. At each time $t \geq 1$, for each $A_{ij}(t-1) \neq 0$, $i \neq j$ we denote the

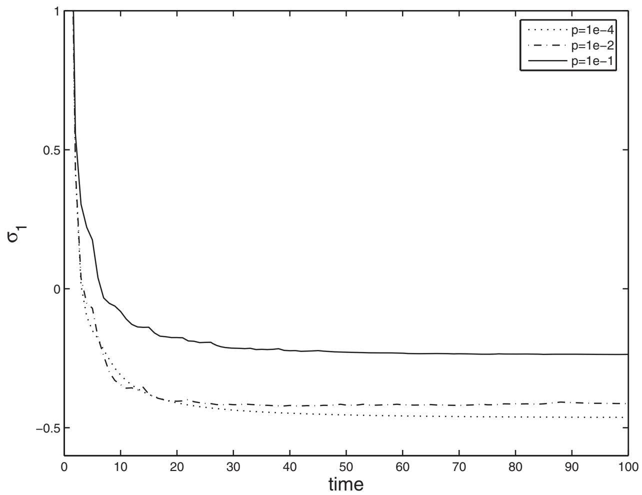

FIG. 1. *Convergence of the second Lyapunov exponent σ_1 for the blinking topology during the topology evolution with the same recovery time $T = 3$ and different malfunction probabilities $p = 10^{-1}$, $p = 10^{-2}$, and $p = 10^{-4}$. The initial scale-free graph is constructed by the method introduced in* [37] *with network size* 500 *and average degree* 12.

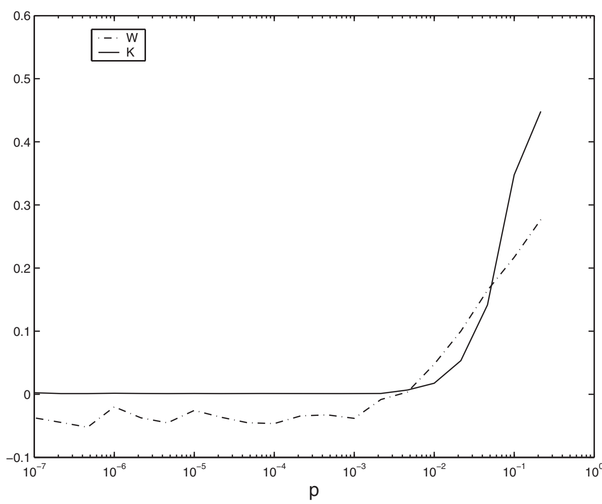

FIG. 2. *Variation of K and W with respect to p for the blinking topology.*

difference $A_{ij}(t) - A_{ij}(t-1)$ by a Gaussian distribution $\mathcal{N}(0, r^2)$ which is statistically independent for all $i \neq j$ and $t \in \mathbb{Z}^+$. If this results in $A_{ij}(t)$ being negative, a weight will be added to the reversal orientation, i.e., $A_{ji}(t) = |A_{ij}(t)|$ and $A_{ij}(t) = 0$. Moreover, if as a result of the process above there exists some index i such that $A_{ij} = 0$ holds for all $j = 1, 2, \ldots, m$, then pick $A_{ii}(t) = 1$.

In Figure 3, we show the convergence of the second Lyapunov exponent σ_1 during the topology evolution for different values of the Gaussian distribution variance r. Picking $r = 0.05$, we show the synchronization of the coupled system (5.1). Let $K(t) = 1/(m-1)\langle\sum_{i=1}^{m}(x^i(t) - \bar{x}(t))^2\rangle_t$, where $\langle\cdot\rangle_t$ denotes the time average from

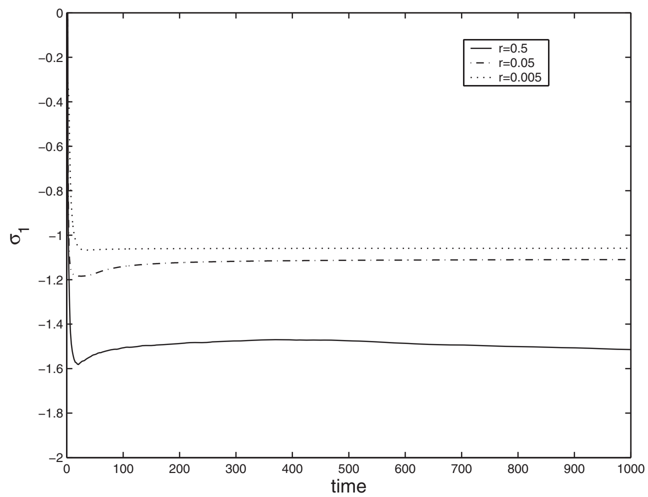

FIG. 3. *Convergence of the second Lyapunov exponent σ_1 for the blurring graph process during the topology evolution with Gaussian variance $r = 0.5$, 0.05, 0.005, and the size of the network $m = 100$.*

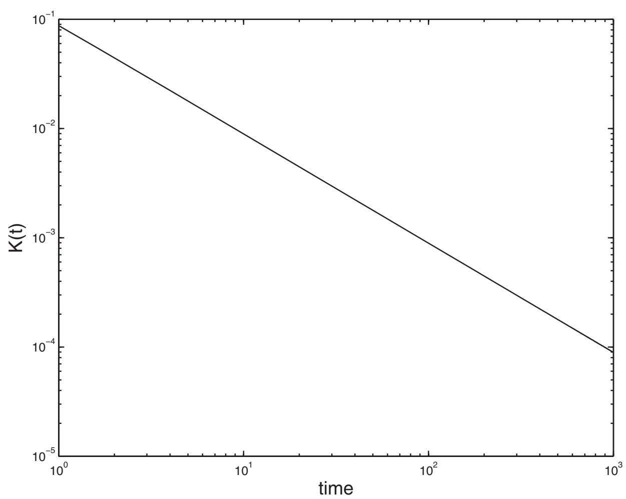

FIG. 4. *Variation of $K(t)$ with respect to time for the blurring graph process.*

0 to t. Since $W = \sigma_1 + \mu$ is about -0.6, i.e., less than zero, the coupled system is synchronized. Figure 4 shows in logarithmic scale the convergence of $K(t)$ to zero.

6. Conclusion. In this paper, we have presented a synchronization analysis for discrete-time dynamical networks with time-varying topologies. We have extended the concept of the Hajnal diameter to generalized matrix sequences to discuss the synchronization of the coupled system. Furthermore, this quantity is equivalent to other widely used quantities such as the projection joint spectral radius and transverse Lyapunov exponents, which we have also extended to the time-varying case. Thus, these results can be used to discuss the synchronization of the CML with time-varying couplings. The Hajnal diameter is utilized to describe synchronizability of the time-varying couplings and obtain a criterion guaranteeing synchronization. Time-varying couplings can be regarded as a stochastic matrix sequence associated with a sequence of graphs. Synchronizability is tightly related to the topology. As we have shown, the statement that $\mathrm{diam}(\mathcal{G}) < 1$, i.e., that chaotic synchronization is possible, is equivalent to saying that there exists an integer T such that the union of the graphs across any time interval of length T has a spanning tree. The methodology will be similarly extended to higher-dimensional maps elsewhere.

Appendix.

Proof of Lemma 2.4. The proof of this lemma comes from [20] with a minor modification. First, we show $\mathrm{diam}(\mathcal{L}, \phi) \leq \hat{\rho}(\mathcal{L}, \phi)$. Let J be any complement of $\mathcal{E}_0$ in $\mathbb{R}^m$ with a basis $u_0, \dots, u_{m-1}$ such that $u_0 = e_0$. Let $A = [u_0, u_1, \dots, u_{m-1}]$, which is nonsingular. Then, for any $t > t_0$ and $t_0 \geq 0$,

$$A^{-1} L_t(\varrho^{(t-t_0)}\phi) A = \begin{bmatrix} c(t) & \alpha_t \\ 0 & \hat{L}_t(\varrho^{(t-t_0)}\phi) \end{bmatrix},$$

where $c(t)$ denotes the row sum of $L_t(\varrho^{(t-t_0)}\phi)$ which is also the eigenvalue corresponding eigenvector e, and $\hat{L}_t(\varrho^{(t-t_0)}\phi)$ can be the solution of linear equation (2.4) with P composed of the rows of A^{-1} except the first row. For any $d > \hat{\rho}(\mathcal{L}, \phi)$, there exists $T > 0$ such that the inequality

$$\left\| \prod_{k=t_0}^{t_0+t-1} \hat{L}_k(\varrho^{(k-t_0)}\phi) \right\| \leq d^t$$

holds for all $t \geq T$ and $t_0 \geq 0$. Let

$$A^{-1} \prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) A = \begin{bmatrix} \prod_{k=t_0}^{t_0+t-1} c(k) & \alpha_t \\ 0 & \prod_{k=t_0}^{t_0+t-1} \hat{L}_k(\varrho^{(k-t_0)}\phi) \end{bmatrix}.$$

Then

$$\begin{aligned} &\left\| A^{-1} \prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) A - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left(\prod_{k=t_0}^{t_0+t-1} c(k), \alpha_t \right) \right\| \\ &= \left\| \begin{bmatrix} 0 & 0 \\ 0 & \prod_{k=t_0}^{t_0+t-1} \hat{L}_k(\varrho^{(k-t_0)}\phi) \end{bmatrix} \right\| \leq C d^t \end{aligned}$$

holds for some constant $C > 0$. Therefore,

$$\left\| \prod_{s=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) - A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left(\prod_{k=t_0}^{t_0+t-1} c(k), \alpha_t \right) A^{-1} \right\| \leq C_1 d^t,$$

$$\left\| \prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) - e \cdot q \right\| \leq C_1 d^t,$$

where $q = \left[\prod_{k=t_0}^{t_0+t-1} c(k), \alpha_t\right] A^{-1}$ and C_1 is a positive constant. It says that all row vectors of $\prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi)$ lie inside the $C_1 d^m$ neighborhood of q. Hence,

$$\mathrm{diam}\left(\prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi)\right) \leq C_2 d^t$$

for some constant $C_2 > 0$, all $t \geq T$, and $t_0 \geq 0$. This implies that $\mathrm{diam}(\mathcal{L}, \phi) \leq d$. Since d is arbitrary, $\mathrm{diam}(\mathcal{L}, \phi) \leq \hat{\rho}(\mathcal{L}, \phi)$ can be concluded.

Second, we show that $\hat{\rho}(\mathcal{L}, \phi) \leq \mathrm{diam}(\mathcal{L}, \phi)$. For any $d > \mathrm{diam}(\mathcal{L}, \phi)$, there exists $T > 0$ such that

$$\mathrm{diam}\left(\prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi)\right) \leq d^t$$

holds for all $t \geq T$ and $t_0 \geq 0$. Letting q be the first row of $\prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi)$, we have

$$\left\|\prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) - e \cdot q\right\| \leq C_3 d^t$$

for some positive constant C_3. Let A be defined as above. Then

$$\left\|A^{-1} \prod_{k=t_0}^{t_0+t-1} L_k(\varrho^{(k-t_0)}\phi) A - A^{-1} e \cdot q A\right\| \leq C_4 d^t,$$

i.e.,

$$\left\|\begin{bmatrix} \prod_{k=t_0}^{t_0+t-1} c(k) & \alpha_t \\ 0 & \prod_{k=t_0}^{t_0+t-1} \hat{L}_k(\varrho^{(k-t_0)}\phi) \end{bmatrix} - \begin{bmatrix} \gamma & \beta \\ 0 & 0 \end{bmatrix}\right\| \leq C_4 d^t$$

holds for some γ and β. This implies that

$$\left\|\prod_{k=t_0}^{t_0+t-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)\right\| \leq C_5 d^t$$

holds for all $t \geq T$, $t_0 \geq 0$, and some $C_5 > 0$. Therefore, $\hat{\rho}(\mathcal{L}, \phi) \leq d$. The proof is completed since d is chosen arbitrarily. ∎

Proof of Lemma 2.6. Let $\hat{\lambda}_{\max} = \sup_{v \in \mathbb{R}^{m-1}} \hat{\lambda}(\mathcal{L}, \phi, v)$. First, it is easy to see that $\log \hat{\rho}(\mathcal{L}, \phi) \geq \hat{\lambda}_{\max}$. We will show $\log \hat{\rho}(\mathcal{L}, \phi) = \hat{\lambda}_{\max}$. Otherwise, there exists $d \in (\exp(\hat{\lambda}_{\max}), \hat{\rho}(\mathcal{L}, \phi))$. By the properties of Lyapunov exponents, for any normalized orthogonal basis $u_1, u_2, \ldots, u_{m-1} \in \mathbb{R}^{m-1}$ with Lyapunov exponent $\hat{\lambda}(\mathcal{L}, \phi, u_i) = \hat{\lambda}_i$, we have, for any $u \in \mathbb{R}^{m-1}$, $\hat{\lambda}(\mathcal{L}, \phi, u) = \hat{\lambda}_{i_u}$, where $i_u \in \{1, 2, \ldots, m-1\}$. $\hat{\rho}(\mathcal{L}, \phi) > d$ implies that there exist $t_0 \geq 0$ and a sequence t_n with $\lim_{n\to\infty} t_n = +\infty$ such that

$$\left\|\prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)\right\| > d^{t_n}$$

for all $n \geq 0$. That is, there also exists a sequence $v_n \in \mathbb{R}^{m-1}$ with $\|v_n\| = 1$ such that

$$\left\| \prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)v_n \right\| > d^{t_n}.$$

There exists a subsequence of v_n (still denoted by v_n) with $\lim_{n\to\infty} v_n = v^*$. Let $\delta v_n = v_n - v^*$. We have

$$\left\| \prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)v^* \right\| \geq \left\| \prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)v_n \right\| - \left\| \prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)\delta v_n \right\|.$$

Note that we can write $\delta v_n = \sum_{i=1}^{m-1} \delta x_n^i u_i$, where $\delta x_n^i \in \mathbb{R}$ with $\lim_{n\to\infty} \delta x_n^i = 0$. So, there exists an integer N such that $\| \prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)\delta v_n\| \leq \big(\sum_{i=1}^{m-1} |\delta x_n^i|\big)d^{t_n}$ holds for all $n \geq N$. Then we have

$$\left\| \prod_{k=t_0}^{t_n+t_0-1} \hat{L}_k(\varrho^{(k-t_0)}\phi)v^* \right\| \geq d^{t_n} - d^{t_n}\left(\sum_{i=1}^{m-1} |\delta x_n^i|\right) \geq C d^{t_n}$$

for all $n \geq N$ and some $C > 0$. This implies $\max_{v\in\mathbb{R}^m} \hat{\lambda}(\mathcal{L}, \phi, v) \geq \log d$, which contradicts the assumption $d \in (\exp(\hat{\lambda}_{\max}), \hat{\rho}(\mathcal{L}, \phi))$. Hence, $\hat{\lambda}_{\max} = \log \hat{\rho}(\mathcal{L}, \phi)$. □

Proof of Lemma 2.7. Recall that $\{\Phi, \mathcal{B}, P, \varrho^{(t)}\}$ denotes a random dynamical system, where Φ denotes the state space, $\mathcal{B}$ denotes the σ-algebra, P denotes the probability measure, and $\varrho^{(t)}$ denotes the semiflow. For a given $\phi \in \Phi$ we denote $L(\varrho^{(t)}\phi)$ by $L(t)$. Let $A = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^{m\times m}$, where $u_1, \ldots, u_m$ denotes a basis of $\mathbb{R}^m$ and $u_1 = e$,

$$A^{-1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^{m\times m}$$

is the inverse of A with

$$\bar{L}(t) = A^{-1}L(t)A = \begin{bmatrix} c(t) & \alpha^\top(t) \\ 0 & \hat{L}(t) \end{bmatrix},\ \hat{L}(t) = A_1^* D(t) A_1,\ \alpha^\top(t) = v_1 L(t) A_1,$$

where $A_1 = [u_2, \ldots, u_m] \in \mathbb{R}^{m\times(m-1)}$, and

$$A_1^* = \begin{bmatrix} v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^{(m-1)\times m}.$$

One can see that the set of Lyapunov exponents of the dynamical system $\{\bar{L}(t)\}_{t\in\mathbb{Z}^+}$ are the same as those of $\{L(t)\}_{t\in\mathbb{Z}^+}$. For any $z(0) = [x(0), y(0)] \in \mathbb{R}^m$, where $x(0) \in \mathbb{R}$ and $y(0) \in \mathbb{R}^{m-1}$, this evolution $z(t+1) = \bar{L}(t)z(t)$ leads to

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c(t-1)x(t-1) + \alpha^\top(t-1)y(t-1) \\ \hat{L}(t-1)y(t-1) \end{bmatrix}.$$

So, we have

$$\text{(A.1)}\quad \begin{aligned} y(t) &= \prod_{k=0}^{t-1}\hat{L}(k)y(0),\\ x(t) &= \prod_{k=0}^{t-1}c(k)x(0)+\sum_{k=1}^{t}\prod_{p=t-k+1}^{t-1}c(p)\alpha^{\top}(t-k)\prod_{q=0}^{t-k-1}\hat{L}(q)y(0). \end{aligned}$$

If the upper bound is less than the lower bound for the left matrix product $\prod$, then the product should be the identity matrix. In the following, we denote by $\hat{\mathcal{L}}$ the projection sequence map of $\mathcal{L}$ and will prove this lemma for two cases.

Case 1. $\lambda_0 \le \log\hat{\rho}(\mathcal{L},\phi)$. Since $\hat{\rho}(\mathcal{L},\phi)$ is just the largest Lyapunov exponent of $\hat{\mathcal{L}}$ defined by $\hat{\lambda}$, from conditions 1 and 2, one can see that for any $\epsilon>0$, there exists $T>0$ such that for any $t\ge T$ it holds that $|\alpha(t)|\le e^{\epsilon t}$, $\|\prod_{k=0}^{t-1}\hat{L}(k)\|\le e^{(\hat{\lambda}+\epsilon)t}$, and $e^{(\lambda_0-\epsilon)t}\le|\prod_{k=0}^{t-1}c(k)|\le e^{(\lambda_0+\epsilon)t}$. Thus, we can obtain

$$\begin{aligned} \prod_{k=t-k+1}^{t-1}|c(p)| &= \prod_{p=0}^{t-1}|c(p)|\times\frac{1}{\prod\limits_{p=0}^{t-k}|c(p)|}\\ &= \begin{cases} e^{(\lambda_0+\epsilon)(t)}e^{-(\lambda_0-\epsilon)(t-k+1)}, & k\le t-T+1,\\ e^{(\lambda_0+\epsilon)(t-1)}\max\limits_{T\ge q\ge 0}\left(\prod\limits_{p=0}^{q}|c(p)|\right)^{-1}, & t-1\ge k\ge t-T. \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} |x(t)| &\le \prod_{k=0}^{t-1}|c(k)||x(0)|+\sum_{k=1}^{t-T+1}\prod_{p=t-k+1}^{t-1}|c(p)||\alpha^{\top}(t-k)|\prod_{q=0}^{t-k-1}\|\hat{L}(q)\|\|y(0)\|\\ &\quad+\sum_{k=t-T}^{t-1}\prod_{p=t-k+1}^{t-1}|c(p)|\|\alpha^{\top}(t-k)\|\prod_{q=0}^{t-k-1}\|\hat{L}(q)\|\|y(0)\|\\ &\le e^{(\lambda_0+\epsilon)t}+\sum_{k=1}^{t-T+1}e^{(\lambda_0+\epsilon)(t-1)}e^{\epsilon t}e^{-(\lambda_0-\epsilon)(t-k)}e^{(\hat{\lambda}+\epsilon)(t-k)}+M_1e^{(\lambda_0+\epsilon)(t-1)}\\ &\le e^{(\hat{\lambda}+\epsilon)t}+e^{(\hat{\lambda}+4\epsilon)t}e^{-(\lambda_0+\epsilon)}\sum_{k=1}^{t-T+1}e^{(-\hat{\lambda}+\lambda_0-3\epsilon)k}+M_1e^{(\lambda_0+\epsilon)t}\\ &\le M_2e^{(\hat{\lambda}+4\epsilon)t}, \end{aligned}$$

where

$$M_1=(T+1)\max_{T\ge q\ge 0}\left(\prod_{p=0}^{q}|c(p)|\right)^{-1}e^{\epsilon T}\left(\prod_{p=0}^{q}\|\hat{L}(p)\|\right)\|y(0)\|,$$

$$M_2=1+M_1+e^{-(\lambda_0+\epsilon)}\sum_{k=1}^{\infty}e^{-3\epsilon k}.$$

So,

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|z(t-1)\|\le\hat{\lambda}+4\epsilon$$

holds for all $z(0)\in\mathbb{R}^m$. Noting that $\hat{\lambda}$ must be less than the largest Lyapunov exponent of $\mathcal{L}$, we conclude that $\hat{\lambda}$ is the largest Lyapunov exponent. This implies the conclusion of the lemma.

Case 2. $\lambda_0>\hat{\lambda}$. Note that for any $\epsilon\in(0,(\lambda_0-\hat{\lambda})/3)$ there exists T such that

$$\prod_{k=0}^{t}|c^{-1}(k)|\|\alpha^\top(t)\|\prod_{l=0}^{t}\|\hat{L}(l)\|\le Ce^{(-\lambda_0+\hat{\lambda}+3\epsilon)t} \tag{A.2}$$

for all $t\ge T$ and some constant $C>0$. Let

$$x=-\sum_{t=0}^{\infty}\prod_{k=0}^{t}c^{-1}(k)\alpha^\top(t)\prod_{l=0}^{t-1}\hat{L}(l)y,$$

which in fact exists and is finite according to inequality (A.2). Then let

$$V_\phi=\left\{z=\begin{bmatrix}x\\y\end{bmatrix}:\ x+\sum_{t=0}^{\infty}\prod_{k=0}^{t}c^{-1}(k)\alpha^\top(t)\prod_{l=0}^{t-1}\hat{L}(l)y=0\right\}$$

be the transverse space. For any $\left[\begin{smallmatrix}x(0)\\y(0)\end{smallmatrix}\right]\in V_\phi$,

$$x(t)=-\sum_{k=t}^{\infty}\prod_{p=t}^{k}c^{-1}(p)\alpha^\top(k)\prod_{q=0}^{k-1}\hat{L}(q)y(0).$$

Noting that there exists $T>0$ such that $\prod_{p=t}^{k}|c^{-1}(p)|\le e^{(-\lambda_0+\epsilon)(k-t)+2\epsilon t}$ for all $t\ge T$, we have

$$\begin{aligned}|x(t)|&\le\sum_{k=t}^{\infty}\prod_{p=t}^{k}|c^{-1}(p)|\|\alpha^\top(k)\|\left\|\prod_{q=0}^{k-1}\hat{L}(q)\right\|\|y(0)\|\\&\le\sum_{k=t}^{\infty}e^{(-\lambda_0+\epsilon)(k-t)}e^{2\epsilon t}e^{\epsilon k}e^{(\hat{\lambda}+\epsilon)k}\\&\le\left\{\sum_{k=t}^{\infty}e^{(-\lambda_0+\hat{\lambda}+3\epsilon)(k-t)}\right\}e^{(\hat{\lambda}+4\epsilon)t}\le M_2e^{(\hat{\lambda}+4\epsilon)t}\end{aligned}$$

for all $t\ge T$and some constants $M_2>0$. So, it can be concluded that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|z(t-1)\|\le\hat{\lambda}+4\epsilon.$$

Since ϵ is chosen arbitrarily, there exists an $(m-1)$-dimensional subspace $V_\phi=\{z=[x\,y]^\top:\ x=-\sum_{t=0}^{\infty}\prod_{k=0}^{t}c^{-1}(k)\alpha^\top(t)\prod_{l=0}^{t-1}\hat{L}(l)y\}$ of which the largest Lyapunov exponent is less than $\hat{\lambda}$. The largest Lyapunov exponent of V_ϕ is clearly greater than $\hat{\lambda}$. Therefore, we conclude that $\hat{\lambda}$, i.e., $\log(\hat{\rho}(L))$, is the largest Lyapunov exponent of L except for λ_0. The proof is completed. ∎

REFERENCES

[1] P. Ashwin, J. Buescu, and I. Stewart, *From attractor to chaotic saddle: A tale of transverse instability*, Nonlinearity, 9 (1996), pp. 703–737.

[2] J. Milnor, *On the concept of attractors*, Commun. Math. Phys., 99 (1985), pp. 177–195.

[3] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge, UK, 2001.

[4] K. Kaneko, *Theory and Applications of Coupled Map Lattices*, Wiley, New York, 1993.

[5] J. Jost and M. P. Joy, *Spectral properties and synchronization in coupled map lattices*, Phys. Rev. E, 65 (2001), article 016201.

[6] Y. H. Chen, G. Rangarajan, and M. Ding, *General stability analysis of synchronized dynamics in coupled systems*, Phys. Rev. E, 67 (2003), article 026209.

[7] F. M. Atay, J. Jost, and A. Wende, *Delays, connection topology, and synchronization of coupled chaotic maps*, Phys. Rev. Lett., 92 (2004), article 144101.

[8] W. Lu and T. Chen, *Synchronization of linearly coupled networks with discrete time systems*, Phys. D, 198 (2004), pp. 148–168.

[9] C. W. Wu, *Synchronization in networks of nonlinear dynamical systems coupled via a directed graph*, Nonlinearity, 18 (2005), pp. 1057–1064.

[10] W. Lu and T. Chen, *Global synchronization of linearly coupled Lipschitz map lattices with a directed graph*, IEEE Trans. Circuits Syst. II Express Briefs, 54 (2007), pp. 136–140.

[11] R. Olfati-Saber and R. M. Murray, *Consensus problems in networks of agents with switching topology and time-delays*, IEEE Trans. Automat. Control, 49 (2004), pp. 1520–1533.

[12] Y. Hatano and M. Mesbahi, *Agreement over random networks*, in Proceedings of the IEEE Conference on Decision and Control, 2004; available online from http://www.aa.washington.edu/faculty/mesbahi/papers/random-cdc.pdf.

[13] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Schochet, *Novel type of phase transitions in a system of self-driven particles*, Phys. Rev. Lett, 75 (1995), pp. 1226–1229.

[14] L. Moreau, *Stability of multiagent systems with time dependent communication links*, IEEE Trans. Automat. Control, 50 (2005), pp. 169–182.

[15] J. H. Lu and G. Chen, *A time-varying complex dynamical network model and its controlled synchronization criterion*, IEEE Trans. Automat. Control, 50 (2005), pp. 841–846.

[16] I. V. Belykh, V. N. Belykh, and M. Hasler, *Connection graph stability method for synchronized coupled chaotic systems*, Phys. D, 195 (2004), pp. 159–187.

[17] D. J. Stilwell, E. M. Bollt, and D. G. Roberson, *Sufficient conditions for fast switching synchronization in time-varying network topologies*, SIAM J. Appl. Dyn. Syst., 5 (2006), pp. 140–156.

[18] J. Hajnal, *Weak ergodicity in nonhomogeneous Markov chains*, Proc. Camb. Phil. Soc., 54 (1958), pp. 233–246.

[19] J. Hajnal, *The ergodic properties of nonhomogeneous finite Markov chains*, Proc. Camb. Phil. Soc., 52 (1956), pp. 67–77.

[20] J. Shen, *A geometric approach to ergodic nonhomogeneous Markov chains*, in Wavelet Analysis and Multiresolution Methods (Urbana-Champaign, IL, 1999), Lecture Notes in Pure and Appl. Math. 212, Dekker, New York, 2000, pp. 341–366.

[21] P. Bohl, *Über Differentialungleichungen*, J. Reine Angew. Math., 144 (1913), pp. 284–313.

[22] L. Barreira and Y. B. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*, University Lecture Series, AMS, Providence, RI, 2001.

[23] F. Colonius and W. Kliemann, *The Lyapunov spectrum of families of time-varying matrices*, Trans. Amer. Math. Soc., 348 (1996), pp. 4398–4408.

[24] J. Wolfwitz, *Products of indecomposable, aperiodic, stochastic matrices*, Proc. Amer. Math. Soc., 14 (1963), pp. 733–737.

[25] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[26] J. Milnor, *On the concept of attractor: Correction and remarks*, Commun. Math. Phys., 102 (1985), pp. 517–519.

[27] I. Daubechies and J. C. Lagarias, *Sets of matrices all infinite product of which converge*, Linear Algebra Appl., 161 (1992), pp. 227–263.

[28] D. Serre, *Matrices: Theory and Applications*, Springer-Verlag, New York, 2002.

[29] F. M. Atay, T. Biyikoglu, and J. Jost, *Network synchronization: Spectral versus statistical properties*, Phys. D, 224 (2006), pp. 35–41.

[30] G. Gripenberg, *Computing the joint spectral radius*, Linear Algebra Appl., 234 (1996), pp. 43–60.

[31] Q. Chen and X. Zhou, *Characterization of joint spectral radius via trace*, Linear Algebra Appl., 315 (2000), pp. 175–188.

[32] X. Zhou, *Estimates for the joint spectral radius*, Appl. Math. Comput., 172 (2006), pp. 332–348.
[33] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Heidelberg, 1998.
[34] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), pp. 197–231.
[35] L. Dieci and E. S. Van Vleck, *Computation of a few Lyapunov exponents for continuous and discrete dynamical systems*, Appl. Numer. Math., 17 (1995), pp. 275–291.
[36] R. Mainieri, *Zeta function for the Lyapunov exponent of a product of random matrices*, Phys. Rev. Lett., 68 (1992), pp. 1965–1968.
[37] A.-L. Barabasi and R. Albert, *Emergence of scaling in random networks*, Science, 286 (1999), pp. 509–512.