

Stability of Coupled Map Networks with Delays*

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Abstract. We consider networks of coupled scalar maps, with weighted connections which may include a time delay, and study the stability of equilibria with respect to the delays and connection structure. We prove that the largest eigenvalue of the graph Laplacian determines the effect of the connection topology on stability. The stability region in the parameter plane shrinks with increasing values of the largest eigenvalue, or of the time delay of the same parity. In particular, all bipartite graphs have an identical stability region, regardless of the delay or graph size, which is also the smallest stability region among those of all graphs. Furthermore, for certain parameter ranges, unstable (and possibly chaotic) maps can be stabilized via diffusive coupling with an odd time delay, provided that the network does not have a nontrivial and connected bipartite component. On the other hand, stabilization is not possible for even values of the delay or for bipartite networks.

Key words. network, delay, stability, synchronization, chaos, Laplacian

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1. Introduction. Coupled map networks were introduced in the 1980s as models of various physical phenomena [1, 2]. Since then, they have become one of the prototype systems for studying collective behavior, such as chaotic synchronization and cluster formation [3]. The interaction between the nodes in such networks is often modeled by a diffusion operator, which intuitively should favor homogenous (i.e., synchronized) behavior of the whole network. Nevertheless, in view of a chaotic system's sensitive dependence on initial conditions, it is a nontrivial finding of the early 1990s that coupled chaotic systems can indeed synchronize [4]. The synchronization of identical chaotic units is rather well understood, and the relevant conditions can be expressed in terms of the largest Lyapunov exponent of the individual maps and the spectrum of the diffusion operator [5]. Because of the diffusive nature of coupling, the synchronized solution is identical to the behavior of individual units in isolation. Thus, a network of chaotic maps will itself be chaotic when synchronized. Partly because of this, the equilibrium solutions of these networks did not receive much attention, although spatially homogeneous equilibria are a special form of synchrony. On the other hand, if one takes into account the time delays in the information transmission between the units, the synchronized dynamics is no longer the same as the isolated behavior [6]. In particular, delays can induce stability in coupled identical limit cycle oscillators [7, 8], leading to the phenomenon

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of amplitude death, which was known earlier for nonidentical oscillators [9, 10]. This suggests the possibility that a synchronized network of chaotic maps might exhibit a stable equilibrium solution if delays are introduced into the model.

In this paper we present a stability analysis of networks of diffusively coupled scalar maps, both with and without time delays. We give necessary and sufficient conditions for stability in terms of the parameters of the individual map and the coupling function, the coupling delay, and the connectivity operator. We show that the role of the coupling topology is characterized solely by the largest eigenvalue of the coupling operator, which provides an order relation for comparing graphs with respect to their stability properties. More precisely, the larger the largest eigenvalue, the smaller the stability region in the parameter space. A similar result has been proved for continuous-time systems near Hopf bifurcation [11]. In the discrete-time setting of this paper, we are able to dispense with the assumption of Hopf bifurcation and extend the result to arbitrary maps. Nevertheless, discrete time has its own peculiarities and introduces some important differences in the stability picture, depending on the parity of the delay. Specifically, we prove that an unstable fixed point of the map can be stable for the network, provided that the delay is odd and that the network has no nontrivial bipartite components, whereas such stabilization is never possible with even (or zero) delays. Moreover, for delays of the same parity, the stability region becomes monotonically smaller as the delay increases, and we calculate the limiting profile of the stability region.

We consider coupled systems of the form

$$(1) \quad x_i(t+1) = f(x_i(t)) + \frac{1}{d_i} \sum_{j=1}^n a_{ij} g(x_i(t), x_j(t-\tau)), \quad i = 1, \dots, n.$$

Here $x_i(t) \in \mathbb{R}$ denotes the state of the i th node at time $t \in \mathbb{Z}$, and $\tau \in \mathbb{Z}^+$ is the signal transmission delay between the nodes. (\mathbb{Z}^+ denotes the nonnegative integers.) The function $f: \mathbb{R} \rightarrow \mathbb{R}$ describes the individual dynamics of the units, and $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is the interaction between a pair of units. We assume that g satisfies the general diffusion condition

$$(2) \quad g(x, x) = 0 \quad \forall x \in \mathbb{R}.$$

The numbers a_{ij} determine the connection structure of the network. In the simplest case, a_{ij} takes on binary values, depending on whether or not there is a connection between the nodes i and j . In other words, $a_{ij} = 1$ if nodes i and j are neighbors, and $a_{ij} = 0$ otherwise, with the stipulation $a_{ij} = a_{ji}$ and $a_{ii} = 0$ for all i, j . (More generally, one might have weighted connections where $a_{ij} \in \mathbb{R}^+$.) The number of neighbors of the i th node is denoted by $d_i = \sum_j a_{ij}$. Disregarding the trivial case of isolated nodes, it is assumed that $d_i > 0$ for all i .

Our interest in this paper is the stability of spatially uniform equilibrium solutions of (1). By virtue of the diffusion condition (2), $\mathbf{x}^* := (x^*, \dots, x^*) \in \mathbb{R}^n$ is a spatially uniform equilibrium solution if and only if $x^* \in \mathbb{R}$ is a fixed point of f . The main question is how the stability of \mathbf{x}^* for the coupled system is related to the stability of x^* for the isolated map. Suppose that $f'(x^*)$ as well as the partial derivatives $D_i g$ of g at (x^*, x^*) exist, and denote

$$b = f'(x^*) \quad \text{and} \quad c = D_2 g(x^*, x^*) = -D_1 g(x^*, x^*),$$

where the last equality is a consequence of (2). Then the linear variational equation about $\mathbf{x}^* = (x^*, \dots, x^*)$ is

$$(3) \quad u_i(t+1) = bu_i(t) + \frac{c}{d_i} \sum_{j=1}^n a_{ij} [u_j(t-\tau) - u_i(t)], \quad i = 1, \dots, n,$$

with $u_i = x_i - x^*$. The asymptotic stability of the zero solution of the linear equation (3) is equivalent to the exponential stability of \mathbf{x}^* in the nonlinear equation (1) (see, e.g., [12]).

In the following sections we present a detailed analysis of the stability of (3). Section 2 derives necessary and sufficient conditions for stability. Section 3 studies the relationship between stability and the delays. Section 4 treats the role of the network topology. In section 5 we look at the implications of the results for the nonlinear system (1) and also mention some extensions to high-dimensional maps. Section 6 concludes the work

2. Stability criteria. We study the stability of the linear system (3). Using the fact that $d_i = \sum_j a_{ij}$, (3) can be written in the vector form

$$(4) \quad \mathbf{u}(t+1) = (b-c)\mathbf{u}(t) + c\mathbf{D}^{-1}\mathbf{A}\mathbf{u}(t-\tau),$$

where $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{A} = [a_{ij}]$, and $\mathbf{D} = \text{diag}\{d_1, \dots, d_n\}$ is a diagonal matrix. Thus, the characteristic values s corresponding to (3) are given by the solutions of

$$(5) \quad \det(s^{\tau+1}\mathbf{I} - s^\tau(b-c)\mathbf{I} - c\mathbf{D}^{-1}\mathbf{A}) = 0.$$

Consequently, the zero solution of (3) is asymptotically stable if and only if all roots s of (5) lie inside the unit circle.

The dynamics of the system (3) is intimately related to the underlying connection structure. Therefore, we shall make use of some graph-theoretical ideas in the analysis. We identify the indices i with the vertices of a graph \mathcal{G} , whose adjacency matrix is $\mathbf{A} = [a_{ij}]$, and the vertex degrees are given by d_i . We assume that \mathcal{G} is a simple and nontrivial graph; i.e., it contains at least one edge and no self-connections. The matrix $\mathbf{D}^{-1}\mathbf{A}$ appearing in (4) depends only on the connection structure of the graph \mathcal{G} . It is related to a (normalized) Laplacian operator \mathbf{L} , which encapsulates the diffusive nature of the coupling. In matrix form, \mathbf{L} is given by

$$(6) \quad \mathbf{L} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}.$$

Although \mathbf{L} is in general not symmetric, it can be shown to be a self-adjoint operator with respect to a certain inner product. We list some relevant spectral properties of \mathbf{L} in the next lemma (see, e.g., [13, 5, 11]). Recall that \mathcal{G} is called a *complete graph* if every vertex is connected to every other vertex, and is called *bipartite* if its vertex set can be divided into two parts V_1 and V_2 such that every edge has one end in V_1 and one in V_2 .

Lemma 1. *Let \mathcal{G} be a graph on n vertices, $n \geq 2$. Then \mathbf{L} is a self-adjoint and positive semidefinite operator; thus its eigenvalues λ_k are real and nonnegative, and its eigenvectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ form a complete orthogonal basis for \mathbb{R}^n . Its smallest eigenvalue is zero and*

corresponds to the eigenvector $(1, 1, \dots, 1)$. If \mathcal{G} has no isolated vertices, then the largest eigenvalue λ_{\max} of \mathbf{L} satisfies

$$(7) \quad \frac{n}{n-1} \leq \lambda_{\max} \leq 2.$$

Furthermore, $\lambda_{\max} = n/(n-1)$ if and only if \mathcal{G} is a complete graph of n vertices, and $\lambda_{\max} = 2$ if and only if a connected component of \mathcal{G} is bipartite and nontrivial.

The lemma implies that for large complete graphs the largest eigenvalue λ_{\max} is close to one. We remark that, if self-connections are included for the nodes, then λ_{\max} for complete graphs is exactly one. At the other extreme are bipartite graphs, for which $\lambda_{\max} = 2$. This is a richer set of graphs, which includes regular lattices in arbitrary dimensions, cycles with an even number of vertices, and all trees. We shall occasionally refer to these two extreme cases in the analysis. The set of eigenvalues of \mathbf{L} is often referred to as the spectrum of the corresponding graph. We note that the spectrum of a graph is the union of the spectra of its connected components.

The system (1) can be generalized in a straightforward way to take into account different connection strengths between pairs of nodes. In this setting, a_{ij} are arbitrary nonnegative numbers denoting the strength of the connection between the i th and j th nodes, where a zero value for a_{ij} indicates that i and j are not connected. The graph \mathcal{G} is now a weighted graph, where the vertex degrees $d_i = \sum_j a_{ij}$ may have noninteger values. We still assume $a_{ij} = a_{ji}$, $a_{ii} = 0$, and $d_i > 0$. The Laplacian is defined by (6), and the spectral properties given in Lemma 1, and in particular (7), remain valid for this more general case [13].

Now using (6) and Lemma 1, one has

$$\mathbf{D}^{-1} \mathbf{A} \mathbf{v}_k = (\mathbf{I} - \mathbf{L}) \mathbf{v}_k = (1 - \lambda_k) \mathbf{v}_k, \quad k = 1, \dots, n,$$

where \mathbf{v}_k and λ_k are the eigenvectors and eigenvalues, respectively, of the Laplacian \mathbf{L} . Hence, in the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $\mathbf{D}^{-1} \mathbf{A}$ can be written as a diagonal matrix, and the characteristic equation (5) can be expressed as the product

$$(8) \quad \prod_{k=1}^n p_k(s) = 0,$$

where

$$(9) \quad p_k(s) := s^{\tau+1} - (b - c)s^\tau - c(1 - \lambda_k).$$

We conclude that the zero solution of (3) is asymptotically stable if and only if all roots of (9) lie inside the unit circle for all $k = 1, \dots, n$.

We first give a sufficient condition for stability that is independent of the delay τ .

Theorem 1. *The zero solution of (3) is asymptotically stable for all $\tau \in \mathbb{Z}^+$ if*

$$(10) \quad |b - c| + |c| < 1,$$

or equivalently,

$$(11) \quad |b| < 1 \quad \text{and} \quad |b - 2c| < 1.$$

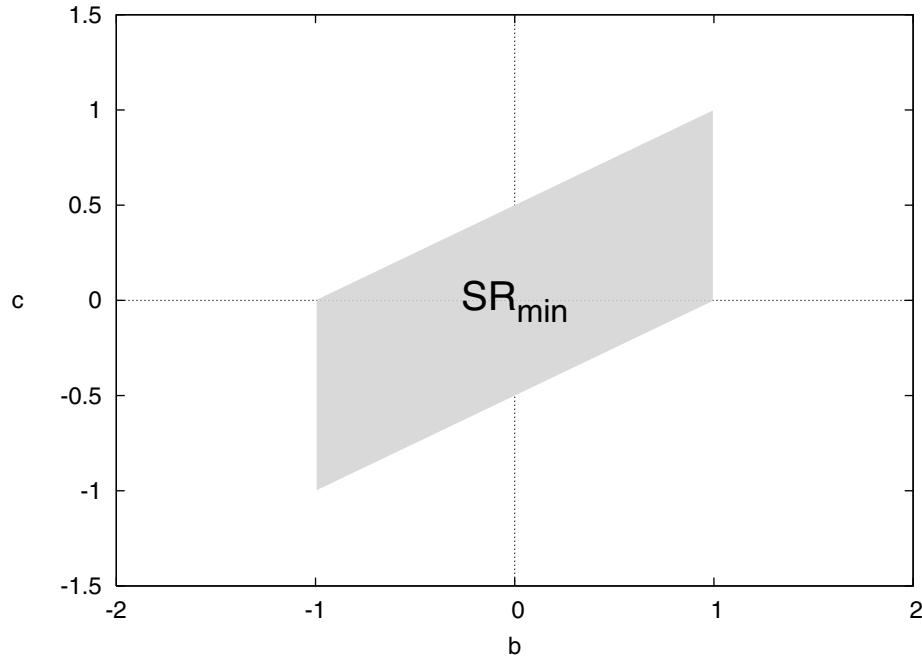


Figure 1. Minimal stability region.

Proof. For $\tau > 0$, a result due to Clark (Lemma 6 in the appendix) implies that the roots of (9) lie inside the unit circle, provided

$$(12) \quad |b - c| + |c(1 - \lambda_k)| < 1.$$

For $\tau = 0$, the characteristic roots are directly solved from (9) as $s = (b - c) + c(1 - \lambda_k)$; so one again has stability if (12) is satisfied. However, (10) implies (12) since $\lambda_k \in [0, 2]$ by Lemma 1, and so (10) is a sufficient condition for stability. The equivalence of (10) and (11) is proved in Lemma 3 in the appendix. ■

In the parameter space (b, c) , we call the set

$$(13) \quad \text{SR}_{\min} = \{(b, c) \in \mathbb{R}^2 : |b - c| + |c| < 1\}$$

$$(14) \quad = \{(b, c) \in \mathbb{R}^2 : |b| < 1 \text{ and } |b - 2c| < 1\}$$

the *minimal stability region*, since by Theorem 1 it is necessarily included in the stability region for any network and any choice of time delay. We shall prove later that SR_{\min} is actually the exact stability region for certain graphs. Figure 1 depicts the shape of the minimal stability region. The bounding lines are

$$(15) \quad c = \frac{b \pm 1}{2} \quad \text{and} \quad b = \pm 1,$$

corresponding to the inequalities in (14).

The precise conditions for stability are given by the next theorem.

Theorem 2. *Let τ be a positive integer. The zero solution of (3) is asymptotically stable if and only if one of the following hold for both $\lambda = 0$ and $\lambda = \lambda_{\max}$ (i.e., for the smallest and the largest eigenvalues of the Laplacian):*

(i) τ is odd and

$$(16) \quad |b - c| - 1 < -c(1 - \lambda) < \sqrt{(b - c)^2 + 1 - 2|b - c| \cos \Phi}.$$

(ii) τ is even,

$$(17) \quad |b - c\lambda| < 1, \quad \text{and}$$

$$(18) \quad |c| < \sqrt{(b - c)^2 + 1 - 2|b - c| \cos \Phi},$$

where Φ is the unique number satisfying

$$(19) \quad \frac{\sin((\tau + 1)\Phi)}{\sin(\tau\Phi)} = |b - c|, \quad \Phi \in \left(0, \frac{\pi}{\tau + 1}\right).$$

For $\tau = 0$, the zero solution is asymptotically stable if and only if (17) holds for $\lambda = 0$ and $\lambda = \lambda_{\max}$.

We will make repeated use of the following simple lemma, whose proof is omitted.

Lemma 2. *Let $p < q$ and $\lambda_1 \leq \lambda_2$ be real numbers, and $h : \mathbb{R} \rightarrow \mathbb{R}$ a monotone¹ function. Then the inequality*

$$p < h(\lambda) < q$$

is satisfied for all $\lambda \in [\lambda_1, \lambda_2]$ if and only if it is satisfied for both $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

Proof of Theorem 2. We first prove sufficiency. Assume that τ is odd and (16) holds for $\lambda = 0$ and $\lambda = \lambda_{\max}$. Then,

$$(20) \quad |b - c| - 1 < -c,$$

$$(21) \quad \text{and } |b - c| - 1 < -c(1 - \lambda_{\max}).$$

If $c \geq 0$, then (20) implies $|b - c| - 1 < 0$, whereas if $c < 0$, then (21) implies $|b - c| - 1 < 0$. In either case we have

$$(22) \quad |b - c| < 1 < \frac{\tau + 1}{\tau}.$$

On the other hand, Lemma 2 implies that (16) holds for all eigenvalues $\lambda_k \in [0, \lambda_{\max}]$ of the Laplacian. We now apply a result of Kuruklis (Lemma 7 in the appendix) with $a_1 = (b - c)$ and $a_2 = -c(1 - \lambda_k)$ and the inequalities (16) and (22), proving that all roots of the polynomials

¹We say $h : \mathbb{R} \rightarrow \mathbb{R}$ is monotone increasing (resp., decreasing) if $h(x_1) \leq h(x_2)$ whenever x_1 is less than (resp., greater than) x_2 . A *monotone* function is one that is either monotone increasing or monotone decreasing on \mathbb{R} .

(9) are within the unit circle for $k = 1, \dots, n$. Next assume that τ is even, and (17) and (18) hold. Since (17) is equivalent to

$$-1 < b - c\lambda < 1$$

and holds for $\lambda = 0$ and $\lambda = \lambda_{\max}$, it holds for all $\lambda \in [0, \lambda_{\max}]$ by Lemma 2. In particular, it holds for $\lambda = 1$, which implies (22). Furthermore, $|c(1 - \lambda)| \leq |c|$ for all $\lambda \in [0, \lambda_{\max}] \subset [0, 2]$. Again, an application of Lemma 7 proves stability.

To prove necessity, assume that all roots of the polynomials (9) are within the unit circle for $k = 1, \dots, n$. If τ is odd, Lemma 7 implies (16) for any $\lambda = \lambda_k$. If τ is even, we use Lemma 7 again to conclude

$$|b - c\lambda_k| < 1,$$

and

$$|c(1 - \lambda_k)| < \sqrt{(b - c)^2 + 1 - 2|b - c| \cos \Phi}$$

for all eigenvalues λ_k of the Laplacian, in particular for the zero and the largest eigenvalues, which imply (17)–(18). On the other hand, when $\tau = 0$, the roots of (9) can be explicitly solved as $s = b - c\lambda_k$. Thus, the zero solution of (3) is asymptotically stable if and only if (17) holds for $\lambda = \lambda_k$, $k = 1, \dots, n$. By Lemma 2, it is necessary and sufficient that this inequality hold for $\lambda = 0$ and $\lambda = \lambda_{\max}$. Finally, the uniqueness of the solution Φ of (19) is proved in Lemma 5 in the appendix. ■

Theorem 2 gives precise conditions for the stability of the coupled system, which will be explored in more detail in the remainder of the paper. The first important observation is that the network topology enters the stability criteria only through the largest eigenvalue λ_{\max} of the Laplacian. The other parameters affecting stability are the time delay τ and the derivatives $b = f'(x^*)$ and $c = D_2g(x^*, x^*)$. For a given λ_{\max} and τ , we denote the stability region in the (b, c) parameter plane as

$$(23) \quad \text{SR}_{\tau, \lambda_{\max}} = \{(b, c) \in \mathbb{R}^2 : \text{the zero solution of (3) is asymptotically stable}\}.$$

In addition to the shape of $\text{SR}_{\tau, \lambda_{\max}}$ in the (b, c) plane, it is of interest how it depends on the time delay and the network structure, as encapsulated by the quantities τ and λ_{\max} , respectively. We give a detailed study of this problem in the following sections.

3. Delays and stability. In this section we investigate how the stability region $\text{SR}_{\tau, \lambda_{\max}}$ defined by (23) depends on the time delay τ . In Theorem 2, the delay affects the stability conditions (16)–(18) implicitly through the quantity Φ defined in (19). In the special case $\tau = 1$ and the limiting case $\tau \rightarrow \infty$, Φ can be explicitly solved and the bounds of the stability region can be analytically determined. Furthermore, the case $\tau = 0$ follows directly from (9), as stated in Theorem 2. For the remaining cases Φ is easy to solve numerically, for instance by the Newton–Raphson method. Figure 2 shows Φ as a function of $|b - c|$ for various values of τ . In the following, we obtain detailed information on how the stability of the system depends on the time delay.

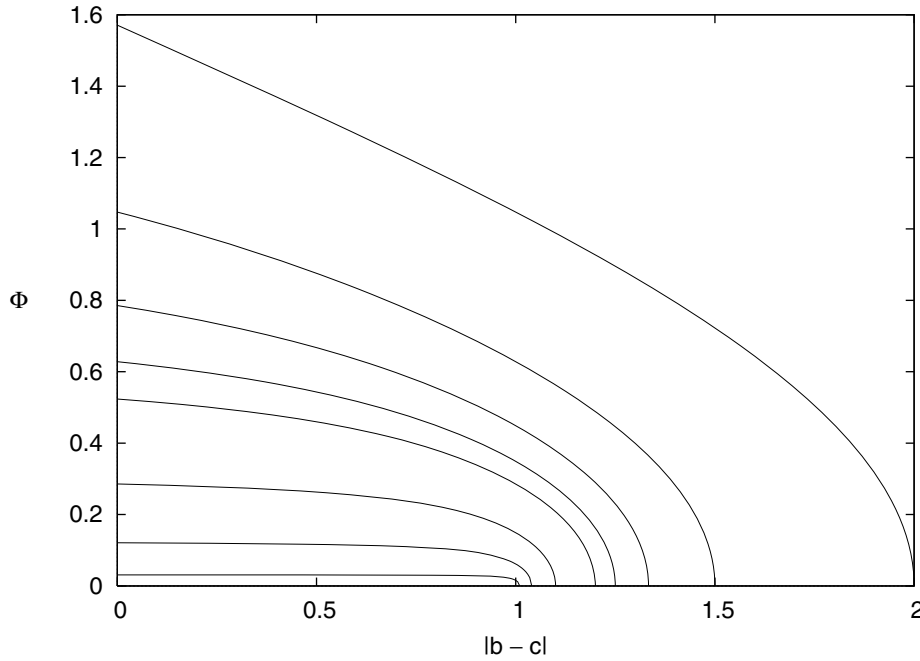


Figure 2. The solution Φ of (19), plotted for τ equal to (from top to bottom) 1, 2, 3, 4, 5, 10, 25, and 100.

3.1. A monotonicity property. We start by proving a monotone dependence on the delay and the limiting behavior for large delays.

Theorem 3. Let $\lambda_{\max} \in [1, 2]$, and let τ_1 and τ_2 be positive integers which are both odd or both even. If $\tau_2 > \tau_1$, then

$$(24) \quad \text{SR}_{\tau_2, \lambda_{\max}} \subset \text{SR}_{\tau_1, \lambda_{\max}}.$$

Furthermore, if $\{\tau_j : j = 1, 2, \dots\}$ is a sequence of even (or odd) integers tending to infinity, then

$$(25) \quad \lim_{j \rightarrow \infty} \text{SR}_{\tau_j, \lambda_{\max}} \subset \text{cl}(\text{SR}_{\min}),$$

where cl denotes closure in \mathbb{R}^2 .

Proof. Suppose that τ_1 and τ_2 are both odd or both even, with $\tau_1 < \tau_2$. We fix a point $(b, c) \in \text{SR}_{\tau_2, \lambda_{\max}}$. We will prove that $(b, c) \in \text{SR}_{\tau_1, \lambda_{\max}}$. To show this, let $\Phi = \Phi(\tau)$ be the solution of (19) belonging to the interval $(0, \pi/(\tau + 1))$. By Lemma 5 in the appendix, Φ is a decreasing function of τ , and so $\cos \Phi$ is increasing in τ . Therefore, the radicands in (16) and (18) are decreasing functions of τ , which implies that (16) or (18) is satisfied for $\Phi(\tau_2)$ whenever it is satisfied for $\Phi(\tau_1)$. This proves (24).

Now let $\{\tau_j\}$ be a sequence of even integers tending to ∞ . By the first part of the theorem, the limit $S = \lim_{j \rightarrow \infty} \text{SR}_{\tau_j, \lambda_{\max}}$ exists and equals

$$S = \bigcap_{j=1}^{\infty} \text{SR}_{\tau_j, \lambda_{\max}}.$$

Hence, if $(b, c) \in S$, then b and c satisfy (17) for $\lambda = 0$ and $\lambda = \lambda_{\max}$ and (18) for all j . Now, (17) is linear in λ , so it also holds for $\lambda = 1$ by Lemma 2, which implies

$$(26) \quad |b - c| - 1 < 0.$$

On the other hand, since $\Phi_j \triangleq \Phi(\tau_j) \in (0, \pi/(\tau_j + 1))$, we have $\cos \Phi_j \rightarrow 1$, and using this in (18) gives

$$(27) \quad |c| \leq ||b - c| - 1|.$$

Combining (27) with (26) yields

$$(28) \quad |c| \leq 1 - |b - c|,$$

which describes the closure of the minimal stability region SR_{\min} given by (13). Hence, $S \subset \text{cl}(\text{SR}_{\min})$, proving (25). A similar argument works also for the case when $\{\tau_j\}$ are odd integers. In this case b and c satisfy (16) as $\cos \Phi \rightarrow 1$; that is,

$$|b - c| - 1 < -c(1 - \lambda) \leq ||b - c| - 1|$$

for $\lambda = 0$ and $\lambda = \lambda_{\max}$, and therefore also for $\lambda = 1$. Evaluating at $\lambda = 1$ gives (26) as above. Evaluating at $\lambda = 0$ and using (26) yields

$$(29) \quad |b - c| - 1 < -c \leq 1 - |b - c|,$$

which implies (28) and proves (25). \blacksquare

By Theorem 1, the set SR_{\min} is contained in every stability region $\text{SR}_{\tau, \lambda_{\max}}$. Hence, Theorem 24 implies that the stability region $\text{SR}_{\tau, \lambda_{\max}}$ approaches SR_{\min} for large delays. Nevertheless, there are some important qualitative differences resulting from the parity of the delay, as we show next.

3.2. Zero delay and even delays. By Theorem 2, the stability region for $\tau = 0$ is given by

$$(30) \quad \text{SR}_{0, \lambda_{\max}} = \{(b, c) : |b| < 1 \text{ and } |b - c\lambda_{\max}| < 1\}.$$

Figure 3 depicts the region $\text{SR}_{0, \lambda_{\max}}$, together with the minimal stability region SR_{\min} . The boundary curves are given by

$$(31) \quad c = \frac{b \pm 1}{\lambda_{\max}} \quad \text{and} \quad b = \pm 1.$$

Note that if $\lambda_{\max}^A > \lambda_{\max}^B$, then

$$\text{SR}_{0, \lambda_{\max}^A} \subsetneq \text{SR}_{0, \lambda_{\max}^B}.$$

Furthermore, comparison of (31) and (15) shows that the stability region of a bipartite graph ($\text{SR}_{0,2}$) coincides with the minimal stability region SR_{\min} . We shall generalize these observations to the case of positive delays in section 4.

The stability regions for other even delays are similar in shape, although they get smaller with increasing τ . Figure 3 shows the stability regions for $\tau = 0$ and $\tau = 2$. In all cases, the stability region is a subset of the strip $|b| < 1$, as follows from Theorem 2 by evaluating (17) at $\lambda = 0$. In other words, an unstable fixed point of the isolated map f cannot be stabilized by any coupling strength or topology when the delay is even.

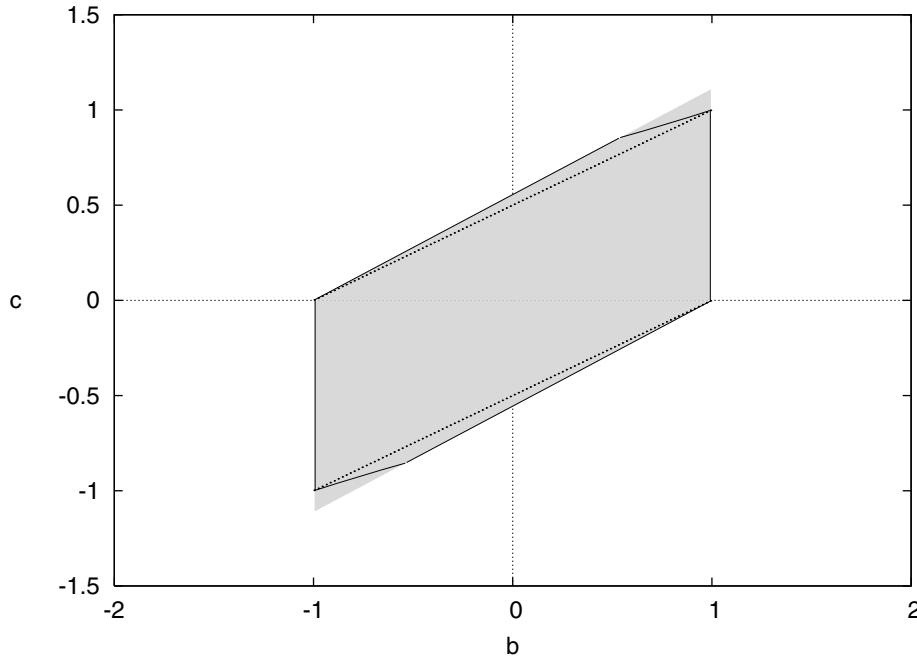


Figure 3. Stability regions for even delays: $\tau = 0$ (gray area) and $\tau = 2$ (inside solid boundaries), plotted for $\lambda_{\max} = 1.8$. The dotted lines delineate the minimal stability region.

3.3. Odd delays. When $\tau = 1$, the equation (19) for Φ yields $\cos \Phi = |b - c|/2$. Substituting into (16), we obtain

$$(32) \quad |b - c| - 1 < -c(1 - \lambda) < 1$$

for $\lambda = 0, \lambda_{\max}$. A straightforward calculation involves these four inequalities in different regions according to the signs of c and $b - c$. The resulting stability region is depicted in Figure 4. For comparison, the stability boundaries for the undelayed case are also shown in the figure. The stability changes introduced by the delay can be summarized as follows. For $c \geq 0$ one easily obtains $|b - c| - 1 < -c$ from (32), which coincides with the upper half of the minimal stability region (13),

$$0 < c < \frac{b + 1}{2}, \quad |b| < 1.$$

For $c < 0$, three different domains can be identified: in the region

$$(1 - \lambda_{\max}) < b < 1, \quad \frac{b - 1}{\lambda_{\max}} < c < 0$$

the stability is unchanged from the undelayed case; for

$$-1 < b < (1 - \lambda_{\max}), \quad -1 < c < 0,$$

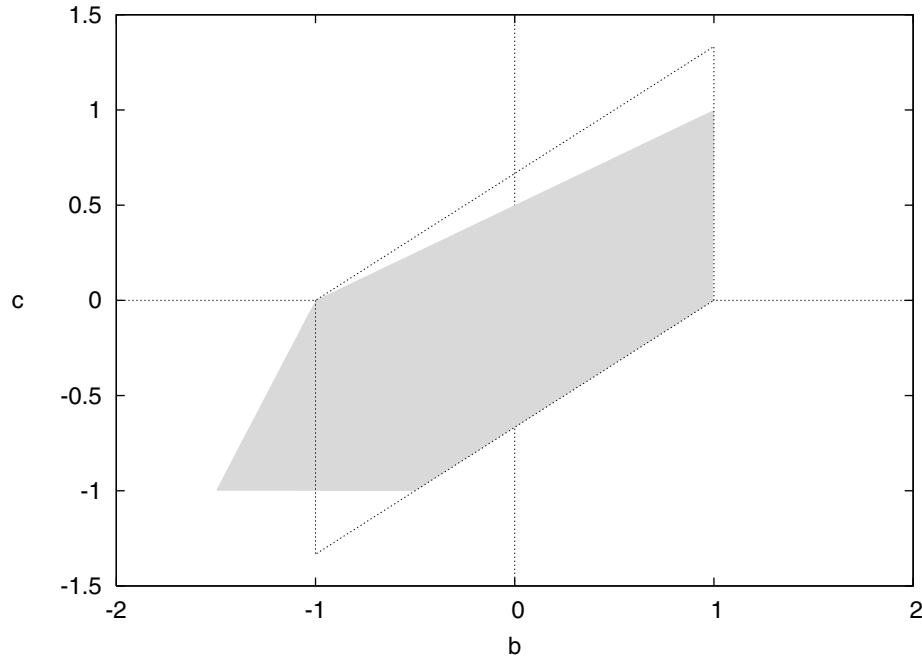


Figure 4. Stability regions for $\tau = 1$ (gray area) and $\tau = 0$ (inside dotted lines). The value of λ_{\max} is 1.5.

it is worsened; and for

$$(33) \quad (\lambda_{\max} - 3) < b < -1, \quad -1 < c < \frac{b+1}{(2 - \lambda_{\max})}$$

it is improved. The last possibility is particularly interesting, as it extends the stability region beyond the strip $|b| < 1$. In other words, an unstable fixed point of f can be stabilized in a network with $\tau = 1$, provided that (33) holds.

The stability region is similar for other odd values of the delay. Figure 5 depicts a close-up view of the region where stability extends into the domain $b < -1$ by the virtue of delays. In accordance with Theorem 3, the stability region gets smaller as the delay gets larger; however, it still extends into $b < -1$. As already mentioned, the stability region is necessarily confined to the strip $|b| < 1$ for even values of the delay. Hence, stabilization of unstable maps by coupling is a feature of odd delays only.

4. Stability and network topology. We now turn to the role of the connection topology on stability. We have already seen that the stability depends on the graph topology only through the value of the largest eigenvalue λ_{\max} of the graph Laplacian. We now prove a monotonicity property; namely, the smaller the value of λ_{\max} , the larger the stability region on the (b, c) -plane for the same value of τ .

Theorem 4. *Let $1 \leq \lambda_A \leq \lambda_B \leq 2$. Then $\text{SR}_{\tau, \lambda_B} \subset \text{SR}_{\tau, \lambda_A}$.*

Proof. Let $(b, c) \in \text{SR}_{\tau, \lambda_B}$. Then by Theorem 2, b and c satisfy either (16) or (17)–(18), depending on whether τ is odd or even, for both $\lambda = 0$ and $\lambda = \lambda_B$. Because these inequalities

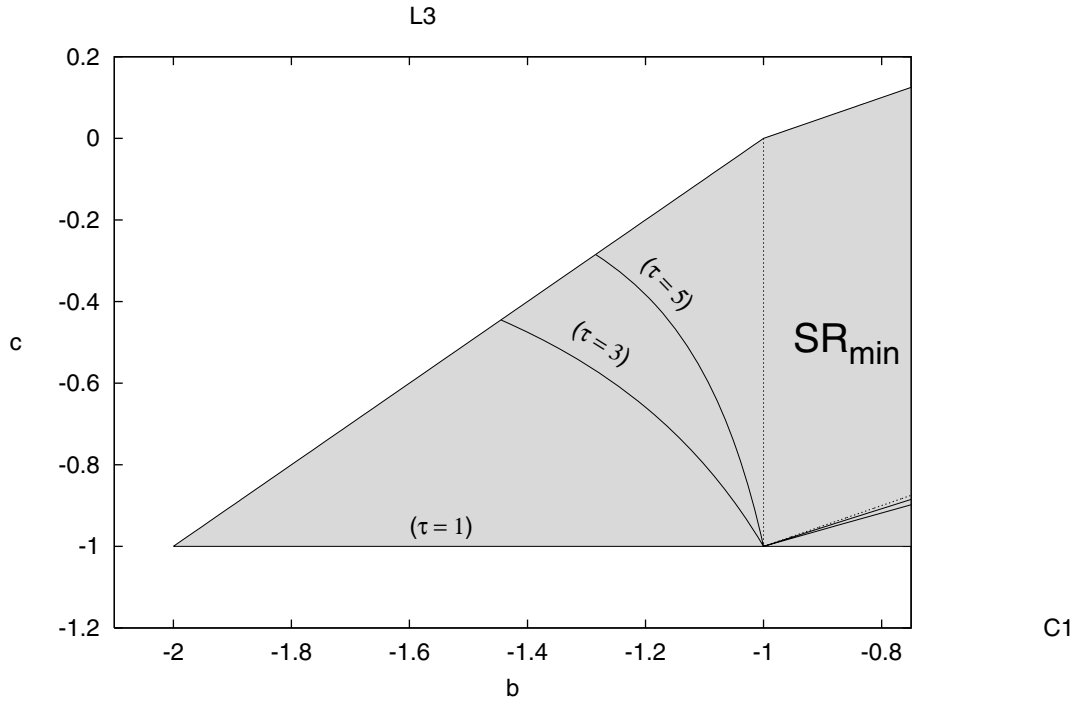


Figure 5. Part of the stability region for odd delays, plotted for $\lambda_{\max} = 1$. As the delay increases, the stability region approaches SR_{\min} .

are linear in λ , they also hold for $\lambda = \lambda_A$, by Lemma 2. Since the conditions (16) or (17)–(18) are also sufficient for stability, $(b, c) \in SR_{\tau, \lambda_A}$. ■

Theorem 4 provides an ordering relation for graphs with respect to their stability properties. Thus, if \mathcal{G}_A and \mathcal{G}_B are two graphs such that $\lambda_A \leq \lambda_B$, where λ_A and λ_B denote the largest eigenvalue of the respective Laplacian, and if $b, c \in \mathbb{R}$ and $\tau \in \mathbb{Z}^+$, then the zero solution of (3) is asymptotically stable for \mathcal{G}_A whenever it is asymptotically stable for \mathcal{G}_B . Figure 6 shows the shrinking of the stability region as λ_{\max} increases.

We next show that all bipartite graphs have the same stability region, which in fact equals the minimal stability region SR_{\min} . An important implication of this property is that the stability region for bipartite graphs is independent of the delay τ . Recall from Lemma 1 that $\lambda_{\max} = 2$ for bipartite graphs.

Theorem 5. $SR_{\tau, 2} = SR_{\min}$ for all $\tau \in \mathbb{Z}^+$. Hence, the stability regions of all bipartite² graphs are identical and independent of the delay τ or the graph size n .

Proof. Let $(b, c) \in SR_{\tau, 2}$. Assume first that τ is odd. By Theorem 2, b and c satisfy

$$(34) \quad |b - c| - 1 < -c(1 - \lambda)$$

for $\lambda = 0$ and $\lambda = 2$. If $c \geq 0$, we use (34) with $\lambda = 0$; otherwise we use it with $\lambda = 2$, to

²More generally, the statement holds for all graphs which have a connected and nontrivial bipartite component.

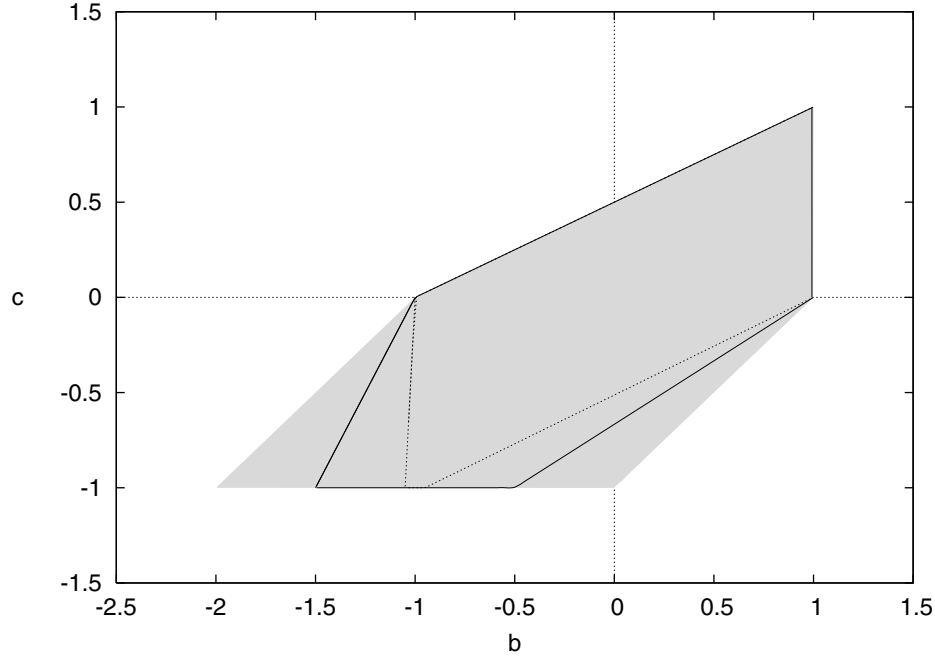


Figure 6. The effect of the network topology: stability regions for $\lambda_{\max} = 1$ (gray area), $\lambda_{\max} = 1.5$ (inside solid lines), and $\lambda_{\max} = 1.95$ (inside dotted lines). As λ_{\max} approaches 2, the stability region approaches SR_{\min} . A value of $\tau = 1$ is used in the plots.

obtain

$$(35) \quad |b - c| - 1 < -|c|.$$

Similarly, for τ positive and even, the condition (17) with $\lambda = 0$ and $\lambda = 2$ gives

$$(36) \quad |b| < 1 \quad \text{and} \quad |b - 2c| < 1.$$

The inequalities in (36) also hold for the case $\tau = 0$, as can be seen by setting $\lambda_{\max} = 2$ in (30). By (14) it follows that $\text{SR}_{\tau,2} \subset \text{SR}_{\min}$. Moreover, $\text{SR}_{\min} \subset \text{SR}_{\tau,2}$ by Theorem 1. Hence, $\text{SR}_{\tau,2} = \text{SR}_{\min}$. ■

Corollary 1. *If $|b| > 1$, then the zero solution of (3) is unstable for a bipartite graph for any choice of parameters. Hence, an unstable fixed point of the map f cannot be stabilized in a bipartite network configuration.*

Proof. By Theorem 5, the stability region of a bipartite graph is SR_{\min} , which is a subset of the strip $|b| < 1$ by (14). It follows that if $|b| > 1$, then the zero solution of the linear equation (3) is unstable. ■

Recalling the spectral properties summarized in Lemma 1, the above results imply that, for a given value of delay τ , a complete graph has the best stability characteristics among all graphs of the same size, and its stability improves slightly with increasing graph size. On the other hand, bipartite graphs have the worst stability characteristics among all graphs of any size and for any value of delay. In fact, a bipartite graph configuration is incapable of

stabilizing an unstable fixed point of the individual map. All other graph configurations lie between the two extremes of bipartite and complete graphs.

5. Nonlinear and higher-dimensional maps. We summarize the implications of the foregoing results for the nonlinear system (1), which we reproduce here for convenience:

$$(37) \quad x_i(t + 1) = f(x_i(t)) + \frac{1}{d_i} \sum_{j=1}^n a_{ij}g(x_i(t), x_j(t - \tau)), \quad i = 1, \dots, n.$$

Furthermore, we briefly discuss some extensions to higher-dimensional maps.

In section 3 we have seen that stabilization is possible only for odd values of the delay. Moreover, Theorem 3 implies that the choice of $\tau = 1$ gives the largest stability region in the (b, c) parameter plane. It follows that the inequalities (33), i.e.,

$$(38) \quad \lambda_{\max} - 3 < f'(x^*) < -1$$

and

$$(39) \quad -1 < c < \frac{f'(x^*) + 1}{(2 - \lambda_{\max})},$$

are necessary conditions for stability in the case $|f'(x^*)| > 1$. From a bifurcation point of view, (38) implies that the network can stabilize x^* in case of a flip bifurcation in the isolated map ($f'(x^*) = -1$) but not in a saddle-node, transcritical, or pitchfork bifurcation ($f'(x^*) = 1$). In particular, an appropriate network configuration can suppress chaotic oscillations originating from a period doubling route to chaos. Furthermore, (38) can be satisfied only if $\lambda_{\max} < 2$, that is, if no nontrivial component of the graph \mathcal{G} is bipartite (viz. Corollary 1). In short, the ingredients of a stabilization scheme involve a flip bifurcation, a nonbipartite network configuration, and an odd coupling delay.

As an example, consider the well-known quadratic family of maps $f_\rho(x) = \rho x(1 - x)$ on the unit interval with parameter $\rho \in [0, 4]$. There are two fixed points, one at zero with $f'_\rho(0) = \rho$, and the other one at $x_\rho^* = (\rho - 1)/\rho$ with $f'(x_\rho^*) = 2 - \rho$. The origin undergoes a transcritical bifurcation at the parameter value $\rho = 1$ and is unstable for $1 < \rho \leq 4$ for the map f_ρ . Hence by the preceding analysis, the origin cannot be stabilized in any network configuration and by any diffusive-type coupling. On the other hand, x_ρ^* loses its stability at $\rho = 3$ through a flip bifurcation and is an unstable fixed point of f_ρ for $3 < \rho \leq 4$. By (38), x_ρ^* can be stabilized for parameter values in the range

$$3 < \rho < 5 - \lambda_{\max}$$

with a choice of c satisfying

$$-1 < c < \frac{3 - \rho}{(2 - \lambda_{\max})}.$$

The largest stability region is provided by large complete graphs, as observed in section 4, where $\lambda_{\max} \rightarrow 1$ as $n \rightarrow \infty$. In this case, one can obtain stability for up to $\rho = 4$, i.e., well

into the fully chaotic regime. (Although the analysis is inconclusive about the case $\rho = 4$, numerical solution with complete graphs having self-connections indicates stability also for the value $\rho = 4$.)

A similar analysis can be extended to m -dimensional maps, i.e., when $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$. In this case, linearizing and expressing in the eigenbasis of the Laplacian leads to the equations

$$(40) \quad \mathbf{u}(t+1) = (\mathbf{B} - \mathbf{C})\mathbf{u}(t) + (1 - \lambda_i)\mathbf{C}\mathbf{u}(t - \tau), \quad i = 1, \dots, n,$$

where $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{m \times m}$, and λ_i are the eigenvalues of the Laplacian as before. Hence, the spatially homogeneous fixed point \mathbf{x}^* is exponentially stable if and only if the zero solution is asymptotically stable in (40) for $i = 1, \dots, n$. The depiction of detailed stability regions will not be attempted here due to the large number of parameters. We confine ourselves to some special cases that allow for some immediate conclusions.

In the classical coupled map lattice model, the interaction function is $g(x, y) = \kappa(f(y) - f(x))$, where the scalar $\kappa \in \mathbb{R}$ plays the role of a coupling strength. Using $\mathbf{C} = \kappa\mathbf{B}$ in (40), and assuming that \mathbf{B} is diagonalizable and has eigenvalues b_j , one sees that \mathbf{x}^* is exponentially stable if and only if all roots of the equation

$$(41) \quad s^{\tau+1} - b_j(1 - \kappa)s^\tau - b_j\kappa(1 - \lambda_i) = 0$$

are inside the unit circle for $i = 1, \dots, n$, $j = 1, \dots, m$. Another common interaction type is linear diffusion, where $g(x, y) = \kappa(y - x)$ for some $\kappa \in \mathbb{R}$, so that $\mathbf{C} = \kappa\mathbf{I}$. With the same assumptions on \mathbf{B} as above, we obtain

$$(42) \quad s^{\tau+1} - (b_j - \kappa)s^\tau - \kappa(1 - \lambda_i) = 0,$$

whose roots determine the stability. In case the eigenvalues b_j are real, both (41) and (42) become special cases of (9), and the results of the previous chapters are applicable.

A special case of (40) is the undelayed network. With $\tau = 0$, (40) reduces to

$$(43) \quad \mathbf{u}(t+1) = (\mathbf{B} - \lambda_i\mathbf{C})\mathbf{u}(t), \quad i = 1, \dots, n,$$

and the stability condition is that all eigenvalues of $\mathbf{B} - \lambda_i\mathbf{C}$, $i = 1, \dots, n$, have modulus less than one. Since the Laplacian always has a zero eigenvalue, a necessary condition for asymptotic stability is that \mathbf{B} have all its eigenvalues inside the unit circle. In other words, unstable fixed points of isolated maps cannot be stabilized in a diffusive network in the absence of delays. This generalizes the corresponding results of section 3. It is interesting to compare (43) to the case obtained by rewriting the delayed system as a higher-dimensional system without delays. Hence, letting $\mathbf{u}(t) = (u(t), u(t-1), \dots, u(t-\tau)) \in \mathbb{R}^{\tau+1}$ in the system (3) of coupled scalar maps, and rewriting in the eigenbasis of the Laplacian, we obtain

$$(44) \quad \mathbf{u}(t+1) = (\mathbf{B} - \mathbf{C}_1 + (1 - \lambda_i)\mathbf{C}_2)\mathbf{u}(t), \quad i = 1, \dots, n,$$

with

$$\mathbf{B} = \begin{pmatrix} b & \mathbf{0}_{1 \times \tau} \\ \mathbf{I}_\tau & \mathbf{0}_{\tau \times 1} \end{pmatrix}, \quad \mathbf{C}_1 = \begin{pmatrix} c & \mathbf{0}_{1 \times \tau} \\ \mathbf{0}_{\tau \times \tau} & \mathbf{0}_{\tau \times 1} \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} \mathbf{0}_{1 \times \tau} & c \\ \mathbf{0}_{\tau \times \tau} & \mathbf{0}_{\tau \times 1} \end{pmatrix},$$

where $\mathbf{0}_{n \times m}$ denotes an $n \times m$ matrix of zeros and \mathbf{I}_τ is the identity matrix of size τ . Note the difference with (43). In (43) the coupling \mathbf{C} acts only through multiplication by λ_i and so has no effect along the eigenvector corresponding to the zero eigenvalue. By contrast, in (44), even when $\lambda_i = 0$ the coupling has a nonzero contribution $\mathbf{C}_2 - \mathbf{C}_1$. This is the essence of the role of delays in stabilizing unstable fixed points.

6. Conclusion. We have presented a general analysis of the stability of equilibrium solutions of diffusively coupled scalar maps, with particular focus on the effects of coupling configuration and delays. These two factors can induce a stability change upon the fixed point of the isolated map, causing a qualitative change in the phase space dynamics. While the loss of stability is not particularly surprising in the presence of delays, the possibility of stabilization by coupling of an otherwise unstable fixed point is an interesting direction in the study of network dynamics. A natural next step is the stabilization of periodic solutions, since these can be expressed as the fixed points of the iterates of the original map. However, the extension requires additional work in the presence of delays, and is deferred to a future publication.

To explore the stabilizing role of delays in more detail, note that the diffusive nature of the interaction, i.e., (2), implies that in the absence of delays the coupling term vanishes whenever the states of the nodes are identical. Consequently, the synchronized dynamics coincides with the isolated dynamics of the map f . This statement is a reflection of the fact that the Laplacian has a zero eigenvalue corresponding to the eigenvector $(1, 1, \dots, 1)$. In other words, diffusive coupling, which helps synchronization by reducing the difference between neighboring nodes, is precisely the reason for the lack of control along the synchronization direction $(1, 1, \dots, 1)$. The situation is markedly different when delays are present. Now the effect of the coupling does *not* vanish along the synchronized direction, which gives it a chance of stabilizing an unstable fixed point of the isolated map. The analysis presented in this paper gives the precise conditions under which such stabilization can occur. We mention in passing that exact synchronization of chaotic maps is still possible under delays, facilitated by the fact that the synchronized dynamics in this case exhibits smaller Lyapunov exponents [6].

The ability of time delays to suppress oscillatory behavior has been observed in continuous-time limit-cycle oscillators [14, 15], where the mechanism essentially involves controlling a Hopf bifurcation. The results of the present paper generalize this effect to networks of arbitrary maps, including chaotic ones, in discrete time. Here it is the flip bifurcation which induces the oscillatory behavior of the map, which is reversed by the collective action of the network, where the parity of the delay has a significant role. Similar to the continuous-time case [11], the effect of the network topology is characterized solely by the largest eigenvalue of the graph Laplacian. On the one hand, the analysis here is more general in the sense that it does not require being near a bifurcation. On the other hand, it assumes scalar units, symmetrical coupling, and a single value for the delay, in contrast to some continuous-time results [11]. The case of multiple delays, especially if they are of different parity, requires different techniques. In case of differential equations, distributed delays are known to be able to enhance stability for single equations [16] as well as networks [17]. To the extent that multiple delays in maps can be considered as the counterpart of distributed delays, one might anticipate further stabilization effects in such general networks; however, this is yet to be

quantified.

Appendix. We prove several intermediate results used in the paper.

Lemma 3. *Let b and c be arbitrary real numbers. Then,*

$$(45) \quad |b - c| + |c| < 1$$

if and only if

$$(46) \quad |b| < 1 \quad \text{and} \quad |b - 2c| < 1.$$

Proof. If (45) holds, then

$$|b| = |b - c + c| \leq |b - c| + |c| < 1$$

and

$$|b - 2c| = |b - c - c| \leq |b - c| + |c| < 1,$$

yielding (46). Similarly, writing (46) as

$$|b - c + c| < 1 \quad \text{and} \quad |b - c - c| < 1$$

and expanding gives

$$-1 - c < b - c < 1 - c,$$

$$-1 + c < b - c < 1 + c.$$

Thus,

$$-1 + |c| < b - c < 1 - |c|,$$

which implies (45). ■

Lemma 4. *Let $p > 1$ and $0 < x \leq 2\pi/(p + 1)$. Then*

$$p \sin x > \sin(px).$$

Proof. Define $g(x) = p \sin x - \sin(px)$. Using the identity

$$\cos(z_1 - z_2) - \cos(z_1 + z_2) = 2 \sin z_1 \sin z_2,$$

we calculate

$$\begin{aligned} g'(x) &= p(\cos x - \cos(px)) \\ &= 2p \sin\left(\frac{p+1}{2}x\right) \sin\left(\frac{p-1}{2}x\right) \\ &> 0, \end{aligned}$$

provided that $0 < x < 2\pi/(p+1)$. Since $g(0) = 0$, it follows that $g(x) > 0$ for $0 < x \leq 2\pi/(p+1)$. ■

Lemma 5. *Let τ be a positive number and $0 \leq a < (\tau+1)/\tau$. Then there exists a unique number $\Phi = \Phi(\tau, a)$ in the interval $(0, \frac{\pi}{\tau+1})$ satisfying*

$$(47) \quad \frac{\sin((\tau+1)\Phi)}{\sin(\tau\Phi)} = a.$$

Furthermore, $\partial\Phi/\partial\tau < 0$.

Proof. We let

$$h_\tau(\Phi) = \frac{\sin((\tau+1)\Phi)}{\sin(\tau\Phi)}.$$

Then $h_\tau : (0, \frac{\pi}{\tau+1}] \rightarrow \mathbb{R}$ is differentiable, and

$$\begin{aligned} h'_\tau(\Phi) &= \frac{(\tau+1)\cos((\tau+1)\Phi)\sin(\tau\Phi) - \tau\sin((\tau+1)\Phi)\cos(\tau\Phi)}{\sin^2(\tau\Phi)} \\ &= \frac{-\tau\sin\Phi + \cos((\tau+1)\Phi)\sin(\tau\Phi)}{\sin^2(\tau\Phi)}. \end{aligned}$$

The sign of $h'_\tau(\Phi)$ is determined by the numerator

$$(48) \quad -\tau\sin\Phi + \cos((\tau+1)\Phi)\sin(\tau\Phi) = \frac{1}{2}[-(2\tau+1)\sin\Phi + \sin((2\tau+1)\Phi)],$$

where we have used the identity

$$2\sin x \cos y = \sin(x+y) + \sin(x-y).$$

We apply Lemma 4 to conclude that the right-hand side of (48) is negative. Thus, h_τ is strictly decreasing over $(0, \frac{\pi}{\tau+1}]$. Furthermore, $h_\tau(\pi/(\tau+1)) = 0$ and

$$\lim_{\Phi \rightarrow 0^+} h_\tau(\Phi) = \frac{\tau+1}{\tau}.$$

Thus, for any a satisfying $0 \leq a < (\tau+1)/\tau$, the equation $h_\tau(\Phi) = a$ has a unique solution, which proves the first statement of the theorem. Now fix $a \in [0, (\tau+1)/\tau)$ and consider (47) for varying τ . Implicit differentiation with respect to τ and rearranging give

$$\begin{aligned} \frac{\partial\Phi}{\partial\tau} &= \Phi \frac{\sin((\tau+1)\Phi)\cos\tau\Phi - \cos((\tau+1)\Phi)\sin\tau\Phi}{(\tau+1)\cos((\tau+1)\Phi)\sin\tau\Phi - \tau\sin((\tau+1)\Phi)\cos\tau\Phi} \\ &= \Phi \frac{\sin\Phi}{-\tau\sin\Phi + \cos((\tau+1)\Phi)\sin\tau\Phi}. \end{aligned}$$

The sign of the last expression is determined by the denominator, which is identical to the left-hand side of (48), which was shown above to be negative. Hence, $\partial\Phi/\partial\tau < 0$. ■

Finally, we list some results from the theory of higher order difference equations related to the study of the characteristic equation

$$(49) \quad s^{k+1} - a_1 s^k + a_2 = 0,$$

where $a_1, a_2 \in \mathbb{R}$ and k is a positive integer. A sufficient condition for stability was given by Clark [18] (see also [19]).

Lemma 6. *All roots of (49) lie inside the unit circle, provided $|a_1| + |a_2| < 1$.*

Necessary and sufficient conditions for the stability of (49) were given by Kuruklis [19].

Lemma 7. *All roots of (49) lie inside the unit circle if and only if*

$$|a_1| < \frac{k+1}{k}$$

and one of the following is satisfied:

$$k \text{ is odd and } |a_1| - 1 < a_2 < (a_1^2 + 1 - 2|a_1| \cos \Phi)^{\frac{1}{2}}$$

or

$$k \text{ is even, } |a_1 - a_2| < 1, \text{ and } |a_2| < (a_1^2 + 1 - 2|a_1| \cos \Phi)^{\frac{1}{2}},$$

where Φ is the unique solution of $\sin((k+1)\Phi)/\sin(k\Phi) = |a_1|$ in the interval $(0, \pi/(k+1))$.

In the statement of the result in [19], a_1 was assumed to be nonnegative; however, the proof given works also without this restriction.

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