Oscillation Control in Delayed Feedback Systems

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Abstract. The control of limit cycle oscillations is discussed for nonlinear delayed feedback systems. Through a center manifold reduction the effects of the control parameters on the periodic behavior are determined. The results are applied to the perturbed harmonic oscillator under the action of delayed feedback of position, which is a prototype system commonly arising in diverse biological and industrial settings. It is shown that, using a cubic feedback function, unwanted periodic solutions can be annihilated, and an asymptotically stable limit cycle can be created oscillating at an arbitrary prescribed amplitude. The frequency of the oscillations can also be controlled to a limited extent. The presence of the delay in the control signal turns out to be crucial for achieving these goals using position feedback .

1 Introduction

Delays in the feedback action is an inevitable feature of many natural and man-made control mechanisms. The presence of the delay usually enriches the possible dynamics and increases the mathematical complexity of the system, even though the undelayed behavior may be deceptively simple. It is a challenging control task to compensate for the destabilizing effects of the delays, or otherwise deal with the added intricacies. On the other hand, it is equally interesting to use the delays to elicit a desired dynamical behavior which may not be possible in an undelayed system.

Positive uses of delays have been noted several decades ago [21], and more recent works have used delays to enhance the system performance in various ways [19,20,18,14]. Traditionally most results in this area seem to focus on linear systems and stability. In the present article, we consider a nonlinear phenomenon, namely limit cycle oscillations , for a class of systems under the action of delayed feedback . Our purpose is to show that the feedback parameters can be so chosen that the controlled system exhibits asymptotically stable periodic oscillations at an arbitrary prescribed amplitude and a range of possible frequencies. Interestingly, the existence of a delay turns out to be a necessary condition for this deed if the feedback uses only position information.

We consider the classical harmonic oscillator and its perturbations controlled by delayed feedback, described by equations of the form

$$\ddot{x} + x + \varepsilon g(x, \dot{x}) = f(x(t-\tau)), \tag{1}$$

where $x \in \mathbf{R}$. Here, g represents the nonlinearities in the system, ε is a nonnegative parameter, and f is the feedback function whose action is delayed by an amount $\tau \geq 0$. Systems of the form (1) arise in various biological and industrial settings, for example in the production of proteins [12,1], orientation control in the fly [17,16], neuromuscular regulation of movement and posture [1,5,7], acousto-optical bistability [22], metal cutting [4], vibration absorption [15], and control of an inverted pendulum [3]. Note that, although the uncontrolled system ($f \equiv 0$) is planar, the closed-loop system is infinite-dimensional if the delay τ is nonzero.

The problem considered in this article is to determine the feedback law f so that (1) has an asymptotically stable periodic solution with desired properties and which attracts all initial conditions in a reasonably large region. Our main result is that this objective can be achieved by using a simple cubic feedback function. Furthermore, the amplitude of the resulting oscillations can be set arbitrarily, while the frequency can be modified to some extent. The conclusion holds for general nonlinearities g and sufficiently small values of ε. Note that f uses only the position information x. Thus, when the derivative $\dot{x}$ is not available for feedback, which is the case in many biological applications, its absence can be compensated for by a suitable delay in the feedback action.

Our strategy can be summarized as follows. Suppose that the uncontrolled system has an equilibrium solution, which we take to be 0 without loss of generality, which is independent of ε. We consider (1) as a perturbation of the linear equation

$$\ddot{x} + x = bx(t - \tau), \tag{2}$$

with ε playing the role of a perturbation parameter. Equation (2) has periodic solutions if it has purely imaginary characteristic values. Such a situation is typical in the study of Hopf bifurcations , where ε is introduced as a scaling factor to blow up some neighborhood of the equilibrium point. In Sect. 3.1 we determine the values of $b, \tau \in \mathbf{R}$ so that (2) has a pair of purely imaginary characteristic values, while the other characteristic values have negative real parts. Thus the long term behavior is governed by the dynamics on the center manifold . In Sect. 3.2 the reduced dynamics on the center manifold is derived for the full nonlinear equation (1). Using the method of averaging, we then investigate the periodic solutions via the fixed points of a scalar equation. Finally, in Sect. 4 we determine a feedback function so that the periodic solutions have a prescribed amplitude and attract all initial conditions in a large domain. While the concept of center manifold reduction and averaging is familiar from the theory of ordinary differential equations, the theoretical and computational issues here are more involved due to the infinite-dimensionality of delayed systems. In the next section we give a brief review of the theory and introduce the notation for the rest of the article.

2 Perturbations of linear retarded equations

Consider the linear system with discrete delays

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - \tau_j), \tag{3}$$

where $x \in \mathbf{R}^n$, $A_j \in \mathbf{R}^{n \times n}$, $\tau_j \geq 0$. Letting $\tau = \max\{\tau_j\}$, an appropriate state-space for (3) is $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbf{R}^n)$, the space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$, equipped with the usual sup norm. Points x_t in $\mathcal{C}$ correspond to "segments" of the solutions x of (3) through $x_t(s) = x(t + s)$ for $s \in [-\tau, 0]$. The characteristic values of (3) are the roots λ of the characteristic equation

$$\det\left(\lambda I - \sum_{j=0}^{m} A_j \exp(-\lambda \tau_j)\right) = 0\,. \tag{4}$$

For each characteristic value λ , there exists a solution of (3) in the form $e^{\lambda t}b$, where b is some constant vector and $t \in \mathbf{R}$. In general (4) has an infinite number of solutions; however, the number of solutions with nonnegative real parts is finite [11].

Associated with (3) is the *adjoint equation*

$$\dot{z}(t) = -\sum_{j=0}^{m} A_j^T z(t+\tau_j), \tag{5}$$

where the superscript T denotes the matrix transpose. Defining $z_t(s) = z(t+s)$ for $s \in [0,\tau]$, the points z_t form a trajectory in $\mathcal{C}^* = \mathcal{C}([0,\tau],\mathbf{R}^n)$ corresponding to the solution z of (5). The spaces $\mathcal{C}$ and $\mathcal{C}^*$ are related through the bilinear form

$$(z_t, x_t) = z_t^T(0)x_t(0) + \sum_{j=0}^{m} \int_0^{\tau_j} z_t^T(s) A_j x_t(s-\tau_j)\, ds\,, \tag{6}$$

so that whenever x and z are solutions of (3) and (5), respectively, which are defined on some interval J, then $(z_t, x_t) \equiv$ constant for all $t \in J$ [13]. The characteristic values of (5) are the roots of the equation

$$\det\left(\lambda I - \sum_{j=0}^{m} A_j^T \exp(-\lambda\tau_j)\right) = 0\,. \tag{7}$$

For each characteristic value λ , there exists a solution of (5) in the form $e^{-\lambda t}b$, where b is some constant vector and $t \in \mathbf{R}$. Note that the characteristic values of (3) and (5) coincide.

Suppose that (3) has d characteristic values (counting multiplicity) with nonnegative real parts, and let Φ be an $(n \times d)$ matrix whose columns form a basis for the generalized eigenspaces corresponding to these characteristic values. Similarly, let Ψ be a matrix whose columns span the generalized eigenspaces of those characteristic values of (5) with nonnegative real parts. It can be shown that the matrix (Ψ,Φ) is nonsingular [9]. Hence, replacing Ψ with $\Psi[(\Psi,\Phi)^{-1}]^T$ if necessary, it may be assumed that (Ψ,Φ) is the identity matrix. Furthermore, there exists a matrix $B \in \mathbf{R}^{d\times d}$, whose eigenvalues are precisely the characteristic values of (3) with nonnegative real parts, such that $\Phi(\theta) = \Phi(0)e^{B\theta}$ for $\theta \in \mathbf{R}$. The bases Φ and Ψ allow a decomposition of the space $\mathcal{C}$: For any $\varphi \in \mathcal{C}$, one has

$$\varphi = \Phi c + \bar{\varphi}\,, \quad c = (\Psi,\varphi)\,.$$

This decomposition is unique for any given φ in $\mathcal{C}$ [10].

Now consider the following perturbation of the linear equation (3)

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-\tau_j) + \varepsilon h(t, x_t), \quad |\varepsilon| \ll 1. \tag{8}$$

where $h : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^n$ is a continuous function. Assume for simplicity that for $\varepsilon = 0$ (8) has no characteristic values with positive real parts. Let x be a solution of (8) with initial value $x_0 \in \mathcal{C}$, and define $\bar{x}_t$ and $y(t) \in \mathbf{R}^d$ by

$$x_t = \Phi y(t) + \bar{x}_t, \quad y(t) = (\Psi, x_t)\,. \tag{9}$$

where Φ and Ψ are as in the previous paragraph. Then y satisfies

$$\dot{y}(t) = By(t) + \varepsilon\Psi^T(0)h(t, \Phi y(t) + \bar{x}_t), \quad y(0) = (\Psi, x_0)\,,$$

where the matrix B has purely imaginary eigenvalues. Furthermore, it can be shown that bounded solutions are such that $\bar{x}_t = O(\varepsilon)$ as $\varepsilon \to 0$ [10]. Hence, to first order in ε, the analysis of (8) is reduced to the investigation of the d-dimensional ordinary differential equation

$$\dot{y}(t) = By(t) + \varepsilon\Psi^T(0)h(t, \Phi y(t)). \tag{10}$$

Since the eigenvalues of B are purely imaginary, the qualitative dynamics are best displayed by changing to polar coordinates and averaging the slowly-varying terms.

3 The harmonic oscillator under delayed feedback

We now return to the harmonic oscillator under the action of delayed feedback

$$\ddot{x} + x = bx(t - \tau)\,, \tag{11}$$

and its perturbations of the form

$$\ddot{x} + x + \varepsilon g(x(t), \dot{x}(t)) = bx(t - \tau) + \varepsilon \bar{f}(x(t - \tau))\,. \tag{12}$$

Using the foregoing ideas, we shall determine the center manifolds of (11), and then obtain the dynamics on the center manifold for the perturbed system (12).

3.1 The linear equation

Letting $(x_1, x_2) = (x, \dot{x})$ puts (11) into the form (3)

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_0 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + A_1 \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}\,, \tag{13}$$

with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\,, \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}\,.$$

The characteristic equation is

$$\chi(\lambda) = \lambda^2 + 1 - be^{-\lambda\tau} = 0\,. \tag{14}$$

The roots $\lambda(b, \tau)$ vary smoothly with the parameters b and τ, and we shall investigate this dependence in order to determine the parameter values for which a two-dimensional center subspace exists. Related studies of (14) have also appeared in [3,6].

If $\lambda = \pm i\omega$ for some $\omega > 0$, it follows from (14) that

$$1 - \omega^2 - b\cos\omega\tau = 0 \tag{15}$$

$$b\sin\omega\tau = 0\,. \tag{16}$$

The possible values of ω are of interest since these give the frequency of periodic solutions. The following lemma summarizes the properties of (14) relevant for the present study. Its proof will also establish the stability diagram given in Fig. 1.

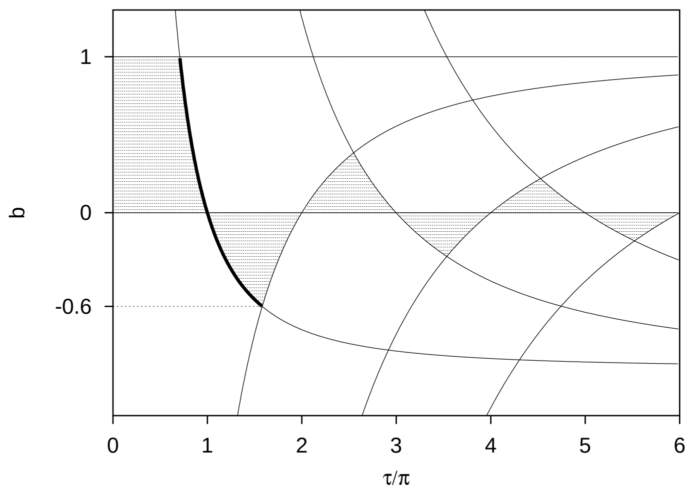

Fig. 1. Regions of stability in the parameter plane

Lemma 1. *Let $\omega \in (\sqrt{2/5}, \sqrt{2})$ and $b = \omega^2 - 1$. Let τ be an arbitrary nonnegative number if $\omega = 1$, otherwise let $\tau = \pi/\omega$. Then the characteristic equation (14) has precisely two roots $\lambda = \pm i\omega$ on the imaginary axis, and no roots with positive real parts.*

Proof. For $b = 0$ and $\tau \in \mathbf{R}$, the only solution to (15)–(16) is $\omega = 1$. When $b \neq 0$, (16) implies that $\omega = \omega_n = n\pi/\tau$ for some nonnegative integer n, and substituting into (15) gives

$$(-1)^n b = 1 - \omega_n^2 = 1 - n^2 \left(\frac{\pi}{\tau}\right)^2. \tag{17}$$

For $n = 0, 1, 2, \ldots$ these equations define a curve γ_n on the b-τ parameter plane, as shown in Fig. 1. Together with the line $b = 0$, they form the set of parameter values for which (14) has a root on the imaginary axis. As these curves are crossed transversally, a pair of roots move across the imaginary axis into the left or the right half plane (except for γ_0, which is the line $b = 1$, where only a single root $\omega = 0$ crosses). The direction of movement is found by implicit differentiation of (14):

$$\frac{\partial \lambda}{\partial b}|_{\lambda = i\omega} = \frac{1}{2\lambda e^{\lambda} + b\tau}|_{\lambda = i\omega} = \frac{1}{(b\tau - 2\omega \sin \omega\tau) + 2i\omega \cos \omega\tau}.$$

Clearly, the sign of the real part of the above expression is determined that by the sign of $(b\tau - 2\omega \sin \omega\tau)$. Thus,

$$\operatorname{sgn}\left(\frac{\partial(\operatorname{Re}\lambda)}{\partial b}|_{\lambda = i\omega}\right) = \begin{cases} -\operatorname{sgn}(\sin\tau) & \text{if } b = 0 \\ \operatorname{sgn}(b) & \text{if } b \neq 0 \end{cases}. \tag{18}$$

It follows that any region of stability must border the $b = 0$ line, since vertically moving away from this line cannot decrease the number of unstable roots. With the knowledge that on the line $b = 0$ there is one pair $\pm i$ on the imaginary axis and no roots on the right half plane, one obtains the regions of stability shown shaded in Fig. 1. On the boundaries of these regions there are purely imaginary roots of the form $\pm i\omega$ and no unstable roots. It is also easy to see that such roots on the imaginary axis are simple. Indeed, if $\chi(\lambda) = 0$, then by (14)

$$\chi'(\lambda) = 2\lambda + b\tau e^{-\lambda\tau} = 2\lambda + \tau(\lambda^2 + 1)$$

Hence $\chi'(i\omega) = 2i\omega + \tau(1 - \omega^2) \neq 0$ whenever $\omega \neq 0$. Therefore, the boundaries of the stability regions, other than γ_0 and the intersections of the γ_i, are the loci of parameter values for which there are precisely two complex conjugate roots on the imaginary axis, and no roots with positive real parts.

To complete the proof, consider the particular boundary segment consisting of the part of the curve γ_1 for which $-3/5 < b < 1$, as shown with a heavy line in Fig. 1. By (17) $b = \omega^2 - 1$ and $\omega = \pi/\tau$ on γ_1; so ω assumes values in the interval $(\sqrt{2/5}, \sqrt{2})$ as the segment is traversed. Therefore, for each parameter pair (τ, b) on the segment there exist precisely two roots $\lambda = \pm i\omega$, except possibly at the intersection with the line $b = 0$. However, it is clear that no additional roots are introduced at the intersection since when $b = 0$ (14) has the unique and unrepeated pair of roots $\lambda = \pm i$.

Now suppose the values of b and τ are chosen as in Lemma 1. A basis for the center subspace of (13) corresponding to the pair of characteristic values $\pm i\omega$ is given by the columns of

$$\Phi = \begin{bmatrix} \cos\omega s & \sin\omega s \\ -\omega\sin\omega s & \omega\cos\omega s \end{bmatrix}, \quad s \in [-\tau, 0]\,. \tag{19}$$

It is easily checked that $\Phi(s) = \Phi(0)\exp(Bs)$, where

$$B = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}. \tag{20}$$

Similarly, the columns of

$$\hat{\Psi} = \begin{bmatrix} \omega\sin\omega s & -\omega\cos\omega s \\ \cos\omega s & \sin\omega s \end{bmatrix}, \quad s \in [0, \tau]$$

span the eigenspace for the adjoint equation for $\lambda = \pm i\omega$. Calculating the bilinear form (6) for the basis vectors and using (15)–(16) gives

$$\begin{aligned} (\hat{\Psi}, \Phi) &= \hat{\Psi}^T(0)\Phi(0) + \int_0^\tau \hat{\Psi}^T(s)A_1\Phi(s-\tau)\,ds \\ &= \begin{bmatrix} \frac{1}{2}\tau(1-\omega^2) & \omega \\ -\omega & \frac{1}{2}\tau(1-\omega^2) \end{bmatrix} \end{aligned}$$

with inverse

$$\frac{4}{\tau^2(1-\omega^2)^2 + 4\omega^2}\begin{bmatrix} \frac{1}{2}\tau\left(1-\omega^2\right) & -\omega \\ \omega & \frac{1}{2}\tau\left(1-\omega^2\right) \end{bmatrix}. \tag{21}$$

Hence if $\hat{\Psi}$ is replaced by $\Psi = \hat{\Psi}[(\hat{\Psi}, \Phi)^{-1}]^T$, one has $(\Psi, \Phi) = I$. Then,

$$\Psi^T(0) = \frac{4}{\tau^2(1-\omega^2)^2 + 4\omega^2}\begin{bmatrix} \omega^2 & \frac{1}{2}\tau\left(1-\omega^2\right) \\ -\frac{1}{2}\omega\tau\left(1-\omega^2\right) & \omega \end{bmatrix}. \tag{22}$$

3.2 The reduced equation and averaging

Consider now the perturbed harmonic oscillator (12), which we rewrite in vector form

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_0 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + A_1 \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ \bar{f}\left(x_1(t-\tau)\right) - g\left(x_1(t), x_2(t)\right) \end{bmatrix}.$$

Using (19), (20), and (22), the reduced equation (10) in this case is calculated as

$$\begin{aligned} \dot{y}(t) &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} y(t) + \varepsilon \Psi^T(0) \begin{bmatrix} 0 \\ \bar{f}\left(\Phi_1(-\tau)y(t)\right) - g\left(\Phi(0)y(t)\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} y(t) + 4\varepsilon \frac{\bar{f}\left(\Phi_1(-\tau)y(t)\right) - g\left(\Phi(0)y(t)\right)}{\tau^2(1-\omega^2)^2 + 4\omega^2} \begin{bmatrix} \frac{1}{2}\tau\left(1-\omega^2\right) \\ \omega \end{bmatrix}, \end{aligned} \tag{23}$$

where Φ_1 denotes the first row of Φ. We switch to polar coordinates

$$y = (r\cos\theta\,, -r\sin\theta)^T \tag{24}$$

in order to put (23) into a form suitable for averaging . Noting that

$$\Phi(-\tau)y = \begin{bmatrix} r\cos\left(\theta - \omega\tau\right) \\ -\omega r \sin\left(\theta - \omega\tau\right) \end{bmatrix}, \quad \text{and} \quad \Phi(0)y = \begin{bmatrix} r\cos\theta \\ -\omega r\sin\theta \end{bmatrix},$$

(23) is transformed into

$$\begin{aligned} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} 0 \\ \omega \end{bmatrix} + \frac{4\varepsilon}{\tau^2(1-\omega^2)^2 + 4\omega^2} \begin{bmatrix} \frac{1}{2}\tau\left(1-\omega^2\right)\cos\theta - \omega\sin\theta \\ -\frac{1}{2r}\tau\left(1-\omega^2\right)\sin\theta - \frac{1}{r}\omega\cos\theta \end{bmatrix} \\ &\quad \times \left(\bar{f}\left(r\cos\left(\theta-\omega\tau\right)\right) - g\left(r\cos\theta, -\omega r\sin\theta\right)\right). \end{aligned} \tag{25}$$

Note that the right hand side is 2π-periodic in θ, and r is slowly varying for small ε. Averaging (25) with respect to θ over one period leads to

$$\dot{r} = -\varepsilon(F(r) + G(r)) \tag{26}$$

$$\dot{\theta} = \omega + \mathcal{O}(\varepsilon)\,, \tag{27}$$

where

$$F(r) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{4\left(\frac{\tau}{2}\left(1-\omega^2\right)\cos\theta - \omega\sin\theta\right)}{\tau^2(1-\omega^2)^2 + 4\omega^2} \bar{f}\left(r\cos\left(\theta-\omega\tau\right)\right)\, d\theta \tag{28}$$

$$G(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{4\left(\frac{\tau}{2}\left(1-\omega^2\right)\cos\theta - \omega\sin\theta\right)}{\tau^2(1-\omega^2)^2 + 4\omega^2} g\left(r\cos\theta, -\omega r\sin\theta\right)\, d\theta\,. \tag{29}$$

Making a change of variables $\theta \mapsto (\theta - \omega\tau)$ in (28), expanding the trigonometric functions, and noting that the integral of any term of the form $\sin\theta \cdot \bar{f}\left(r\cos\theta\right)$ is zero, $F(r)$ can be written as

$$F(r) = \gamma\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, \bar{f}\left(r\cos\theta\right)\, d\theta\,, \tag{30}$$

with

$$\gamma = \frac{4\left(\omega\sin\omega\tau - \frac{1}{2}\tau\left(1-\omega^2\right)\right)}{\tau^2(1-\omega^2)^2+4\omega^2}. \tag{31}$$

Clearly, the quantity γ depends only on the parameters b and τ appearing in the unperturbed linear equation, while the rest of the expression in (30) depends only on the feedback perturbation $\bar{f}$. Hence, $\bar{f}$ can influence the dynamics of (26) only if $\gamma \neq 0$. The next lemma shows that γ is nonzero for almost all values of b, τ under consideration.

Lemma 2. *Suppose b and τ are chosen as in Lemma 1. Then $\gamma = 0$ if and only if $b = 0$ and $\sin\tau = 0$.*

Proof. If $b = (\omega^2-1) \neq 0$ and $\tau = \pi/\omega$, then

$$\gamma = \frac{1}{2}\frac{\pi\omega(\omega^2-1)}{\pi^2(1-\omega^2)^2+4\omega^4} \neq 0,$$

whereas if $b = 0$, then $\omega = 1$ by (15), so that $\gamma = \sin\tau$.

On the other hand, it is obvious from (31) that $\gamma = 0$ if $\tau = 0$, so that the presence of a positive delay is a necessary condition for controlling the dynamics on the center manifold by position feedback.

4 Controlling the amplitude and frequency of oscillations

The averaged equations (26)–(27) contain important information about the periodic solutions of (1). Indeed, by the averaging theorem [8,10] and through (9) and (24), a hyperbolic equilibrium $R > 0$ of (26) corresponds to a nontrivial hyperbolic periodic solution of (1) whose amplitude is near R and frequency is near ω. The problem to address now is how to choose the control f so that the averaged equation (26) has desired dynamics. In particular, we would like to ensure that (26) has a unique equilibrium point at R in some bounded (but large) interval, which attracts all initial conditions there. Our main result shows that this can be achieved by a cubic feedback function.

Theorem 1. *Suppose $g : \mathbf{R}^2 \to \mathbf{R}$ is a C^2 function such that $g(0,0) = 0$. Let $R > 0$ and $\omega \in (\sqrt{2/5}, \sqrt{2})$. Then there exist $\tau > 0$, a feedback function of the form*

$$f(x) = bx + \varepsilon(b_1x + b_3x^3)\,, \tag{32}$$

and $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, (1) has an attracting periodic solution $x(t) = R\cos(\omega t) + \mathcal{O}(\varepsilon)$.

Proof. Given $\omega \in (\sqrt{2/5}, \sqrt{2})$, choose b and τ as in Lemma 1. Using the notation of (12), we have $\bar{f}(x) = b_1x + b_3x$, and the corresponding averaged function F is calculated through (30) as

$$F(r) = q_1r + q_3r^3,$$

with

$$q_1 = \frac{1}{2}\gamma b_1 \quad \text{and} \quad q_3 = \frac{3}{8}\gamma b_3\,. \tag{33}$$

Let G be given by (29), and define

$$\bar{G}(r) = \begin{cases} G(r)/r & \text{if } r \neq 0, \\ G'(0) & \text{if } r = 0, \end{cases} \tag{34}$$

Consequently,

$$F(r) + G(r) = r(q_1 + q_3 r^2 + \bar{G}(r)), \tag{35}$$

Now let $R > 0$ and fix $c > R$. By the assumptions on g it is easy to see that $\bar{G}$ is continuously differentiable on $\mathbf{R}$; so, the following nonnegative numbers exist:

$$\mu_1 = \max\{|\bar{G}(r)| : r \in [0, c]\}, \tag{36}$$
$$\mu_2 = \max\{|\bar{G}'(r)| : r \in [0, c]\}. \tag{37}$$

Choose q_1 such that

$$q_1 < -\frac{5}{3}\mu_1 - R\mu_2, \tag{38}$$

and define q_3 by

$$q_3 = \frac{-q_1 - \bar{G}(R)}{R^2} > 0. \tag{39}$$

Note from (35) that positive roots of $F + G$ and of the function

$$H(r) := q_1 + q_3 r^2 + \bar{G}(r)$$

coincide. Using (39), one has

$$H(r) = q_1 \left(1 - \frac{r^2}{R^2}\right) + \bar{G}(r) - \frac{r^2}{R^2}\bar{G}(R), \tag{40}$$

so, $H(R) = 0$. We claim that H has no other positive roots in $(0, c)$. Indeed, for $r \in (0, R/2]$:

$$\begin{aligned} H(r) &\leq q_1 \left(1 - \frac{r^2}{R^2}\right) + \mu_1 \left(1 + \frac{r^2}{R^2}\right) \\ &\leq \frac{3}{4}q_1 + \frac{5}{4}\mu_1 < -\frac{3}{4}R\mu_2 \leq 0, \end{aligned}$$

where the last inequality follows by (38). Hence, $H(r)$ has no roots on the interval $(0, R/2]$. On the other hand, for $r \in [R/2, c)$,

$$\begin{aligned} H'(r) &= \frac{-2(q_1 + \bar{G}(R))}{R^2} r + \bar{G}'(r) \geq \frac{-2(q_1 + \mu_1)}{R^2}\frac{R}{2} - \mu_2 \\ &\geq \frac{1}{R}\left(-q_1 - \frac{5}{3}\mu_1 - R\mu_2\right) > 0, \end{aligned} \tag{41}$$

Thus H is strictly increasing for $r \in [R/2, c]$, so that any root in this interval is necessarily unique. Hence, R is the unique root of H (and consequently of $F + G$) in the interval $(0, c)$. It remains to show that R is attracting in $(0, c)$. Furthermore,

$$F'(R) + G'(R) = H(R) + RH'(R) = 0 + RH'(R) > 0$$

by (41), and

$$F'(0) + G'(0) = H(0) = q_1 + \bar{G}(0) \leq q_1 + \mu_1 < 0,$$

by (35) and (38). So, in the scalar equation (26) the origin is unstable and R is asymptotically stable. It then follows by the averaging theorem [8,10] that, for all sufficiently small and positive values of ε, (1) has an asymptotically stable periodic solution of the form $x(t) = R\cos\omega t + \mathcal{O}(\varepsilon)$. Since $\gamma \neq 0$ by Lemma 2, the coefficients of (32) are found from (33) as $b_1 = 2q_1/\gamma$ and $b_3 = 8q_3/(3\gamma)$.

Remark. It follows by Lemma 1 and the above proof that if $\omega = 1$, then for any τ satisfying $\sin\tau \neq 0$ a feedback function of the form (32) can be found with the same conclusion as in the statement of the theorem.

Theorem 1 shows that delayed position feedback offers complete control over the amplitude of the periodic solutions and only limited control over the frequency. The case when frequency need not be modified ($\omega = 1$) is particularly interesting since the cubic feedback scheme (32) then works for almost all values of the delay, as remarked above. In addition, the design of the feedback law can then be reduced to the tuning of a single parameter b_1, using the idea of the proof of the theorem. Indeed, by (33) and (39), the coefficient b_3 can be determined from the knowledge of b_1:

$$b_3 = -\frac{8}{3R^2}\left(\frac{1}{2}b_1 + \frac{G(R)}{\gamma R}\right), \tag{42}$$

with $\gamma = \sin\tau$. To obtain oscillations at an amplitude of R, by (38) it is sufficient (but not necessary) to choose b_1 such that γb_1 is large and negative. We present some examples.

Example. For the van der Pol's oscillator under linear feedback

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = \varepsilon b_1 x(t - \tau), \tag{43}$$

the function $G(r) = \frac{1}{8}r\left(r^2 - 4\right)$; so for small ε the uncontrolled system has the familiar limit cycle solution close to $2\cos t$. Putting $\omega = 1$ in (30), one has $F(r) = \frac{1}{2}b_1\sin\tau$, and the averaged equation (26) takes the form

$$\dot{r} = -\frac{1}{8}\varepsilon r(r^2 - 4(1 - b_1\sin\tau)).$$

Choosing $b_1\sin\tau < 1$, the amplitude of the limit cycle is changed to $2\sqrt{1 - b_1\sin\tau}$. On the other hand, choosing $b_1\sin\tau > 1$ destroys the limit cycle and stabilizes the zero solution. Both these feats can be accomplished for any value of the delay satisfying $\sin\tau \neq 0$. For more details the reader is referred to [2].

Example. Consider the system

$$\ddot{x} - \varepsilon\sin(\dot{x}) + x = f(x(t - \tau)). \tag{44}$$

Here, the function $G(r)$ has infinitely many zeros, and thus the uncontrolled system has an infinite number of (stable and unstable) limit cycles, and the origin is unstable. Suppose it is desired to destroy these limit cycles and have an attracting stable

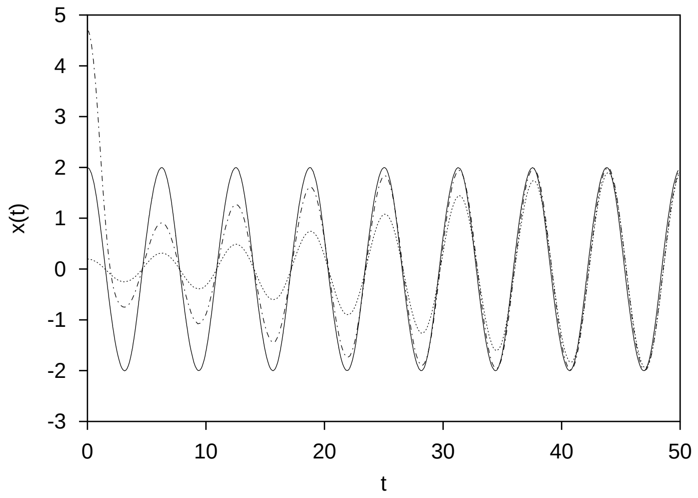

Fig. 2. The limit cycle with amplitude 2, shown with the solid line, obtained for the parameter values $\varepsilon = 0.1$ and $\tau = \pi/2$ with the cubic feedback (32). The dotted lines are trajectories starting from arbitrary initial conditions and converging to the limit cycle

limit cycle oscillating at an amplitude of $R = 2$ and frequency $\omega = 1$. A numerical calculation gives $G(2) = -0.577$. Assuming a delay value of $\tau = \pi/2$, we have $\gamma = 1$ by (31). Choosing $b = 0$ and $b_1 = -0.5$, and determining $b_3 = 0.36$ from (42), the cubic feedback (32) gives the limit cycle shown in Fig. 2. Other trajectories starting from a variety of initial conditions converge to the limit cycle, shown by dotted lines in the same figure. Hence, the domain of attraction of the limit cycle is reasonably large. The extent of the domain depends in general on the values of b_1, ε, and τ, and increases with decreasing ε when other parameters are fixed.

5 Conclusion

It is seen that nonlinear delayed feedback of position can be effective in the control of oscillations. By using a simple cubic feedback function, it is possible to suppress unwanted oscillations or create a limit cycle with a desired amplitude. These observations may help explain the mechanism for the observed oscillatory behavior in biological systems, or design controllers in the absence of derivative information. It thus seems worthwhile to generalize the results presented here to more general classes of systems, which may be higher dimensional, or may contain more general (e.g. distributed) delays. Since our results are largely independent of the particular nonlinearities in the system, or in some cases the actual value of the delay, the proposed feedback scheme may also carry certain robustness properties, which need to be investigated further. The delayed feedback thus presents an interesting and challenging study in the area of nonlinear control.

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