Almost Empty Monochromatic Triangles in Planar Point Sets

Deepan Basu¹

Bhaswar Bikram Bhattacharya²

Sandip Das¹

¹Indian Statistical Institute, Kolkata, India
²Stanford University, California, USA

20 June, 2012
In 1978 Erdős asked whether for every positive integer $k$, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains $k$ points which lie on the vertices of a convex polygon whose interior contains no points of the set.

Such a subset is called an empty convex $k$-gon or a $k$-hole.

Esther Klein showed that $H(4) = 5$ and Harborth proved that $H(5) = 10$. Horton showed that $H(7) = \infty$. Recently Gerken and Nicolás independently proved that $H(6) < \infty$. 
“Almost” empty convex polygons

- The problem of obtaining almost empty convex polygons, that is convex polygons with few interior points, was studied by Nyklova.

- Let $H(k, s)$ be the minimum positive integer such that any set $S$ of at least $H(k, s)$ points in the plane, contains a subset $Z$ whose elements are on the vertices of a convex $k$-gon and there are at most $s$ points of $S$ in the interior of the convex hull of $Z$.

- Nyklova showed that $H(6, 6) = 17, H(6, 5) = 19$ and $H(k, s)$ does not exist for every $s \leq 2^{(k+6)/4} - k - 3$. 
Devillers et al introduced a colored variant, in which the set of points is partitioned into \( r \geq 2 \) color classes, and a convex polygon is said to be monochromatic if all its vertices belong to the same color class.

Grima et al showed that 9 points are necessary and sufficient for a bi-colored point set to have a monochromatic empty triangle.

Devillers et al also constructed specific colored Horton sets to prove that there exists arbitrarily large 3-colored point sets with no monochromatic empty triangle.
They also showed that there exists arbitrarily large 2-colored point sets with no monochromatic 5-hole.

It was conjectured by Devillers et al. that any sufficiently large set of bi-colored points in the plane, no three on a line, contains a monochromatic 4-hole.

However, in spite of substantial attempts over the last few years, the problem remains open. A weaker version which arose by relaxing the convexity condition was proved by Aichholzer et al.
In this paper, we amalgamate these two ideas and study the existence of monochromatic empty triangles with few interior points (or stains) in colored point sets.

For positive integers $c, s \geq 1$ we define $M_3(c, s)$ to be the least integer such that any set of at least $M_3(c, s)$ points in the plane, and colored with $c$ colors, contains a monochromatic triangle with at most $s$ interior points.

If there exists arbitrarily large sets colored with $c$ colors which does not contain a monochromatic triangle with at most $s$ interior points we define $M_3(c, s) = \infty$.
Our Contribution

The case $s = 0$ has been studied extensively over the last few years. We already mentioned that $M_3(1, 0) = 3$, $M_3(2, 0) = 9$ and $M_3(c, 0) = \infty$, for $c \geq 3$

In this paper we extend these results when $c \geq 4$ and $s \geq 1$. We will also show that $M_3(2, 1) = 6$ and $M_3(3, 1) = 13$

We define $\lambda_3(c)$ as the least integer with $M_3(c, \lambda_3(c)) < \infty$. In other words, given a fixed number of colors $c$, $\lambda_3(c)$ is the least integer such that any sufficiently large $c$-colored point set contains a monochromatic triangle with at most $\lambda_3(c)$ interior points.
Main Theorem

**Theorem:** \( \left\lfloor \frac{c-1}{2} \right\rfloor \leq \lambda_3(c) \leq c - 2 \) where \( c \geq 2 \).

For the upper bound we show that in fact \( M_3(c, c - 2) \leq (c + 1)^2 \).

For the lower bound we will construct a certain coloring of Horton set which will prove that \( M_3(2q + 1, q - 1) = \infty \) for any \( q \geq 1 \).
We prove that $M_3(c, c - 2) \leq (c + 1)^2$ by Induction.

The key two steps are:

- A lemma which states $M_3(c, c - 1) \leq c(c + 1) + 1$.
- For any set $S$ of points in the plane in general position, the number of triangles in any triangulation of $S$ is $2|S| - |CH(S)| - 2$.

Using these two arguments, we look at the convex hull of most frequent color in the set and show that there exists such a triangle of either that or the second most frequent color.
Proving Lower bound

- A Horton Set is a set sorted by $x$-coordinates $h_1 < x h_2 < x \cdots < x h_n$ such that any line through two ‘even’ points leaves all ‘odd’ points below and any line through two ‘odd’ points leaves all ‘even’ points above.

- We exploit the recursive structure of Horton Set, as done for most counterexamples in such cases.

- We color the Horton set using $c = 2q + 1$ colors alternately. This restricts the shapes of monochromatic triangles in a favorable way.
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<tr>
<td>There are several interesting problems arising out of this study.</td>
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<tr>
<td>- Obtaining a general lower bound on $M_3(c, c-2)$</td>
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<td>- Improving the bounds on $\lambda_3(c)$</td>
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<td>- Proving Conjecture $\lambda_3(4) = 1$</td>
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<td>- Generalizing these results to monochromatic convex $r$-gons</td>
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<td>Thanks to SIAM Committee and all the organisers!</td>
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