

ON THE MAXIMUM PRINCIPLE FOR LOCALLY VANISHING ELLIPTIC OPERATORS

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(Communicated by Walter Craig)

ABSTRACT. As was shown in [7], for the validity of the classic strong maximum principle for elliptic partial differential equations it is sufficient that the coefficients matrix $a^{ij}(x)$ be positive semi-definite. Recently, in [10], a generalisation of this maximum principle was derived for operators in divergence form, requiring $a^{ij} \in L^\infty(\Omega)$ to be strictly elliptic. In this note we demonstrate that the strict ellipticity condition on $a^{ij}(x)$ is essential and cannot be weakened.

1. INTRODUCTION

The weak maximum principle for classical solutions of second order linear elliptic partial differential equations also holds true if the coefficients matrix is non-negative, see [7, Section 3.1] and the short review below. In this article we show that the corresponding weak maximum principle for weak (sub-)solutions of elliptic partial differential equations in divergence form does *not* allow a generalisation to non-negative coefficients matrices. We do so by constructing a counter example which arises in the context of some non-convex variational problems describing microstructures in crystals.

First we fix the notations and briefly review some weak maximum principles being related to our result. Our counter example involves the highest order term of general second-order linear elliptic partial differential equation only. Therefore we restrict the presentation of the weak maximum principles to the highest order term in second-order linear elliptic partial differential equations. We refer to [7, 3, 9] for the treatment of more general cases.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f(x) \in L^p(\Omega)$ and $a^{ij}(x) \in L^\infty(\Omega)$, $1 \leq i, j \leq n$ such that $a^{ij} = a^{ji}$ and uniformly for almost every $x \in \Omega$

$$(1.1) \quad \lambda \text{Id} \leq (a^{ij}(x))_{1 \leq i, j \leq n} \leq \Lambda \text{Id},$$

where Id denotes the identity matrix in $\mathbb{R}^{n \times n}$ and $0 \leq \lambda \leq \Lambda$ are two real numbers. In case of $\lambda > 0$, this ensures the strict ellipticity of the partial differential equation

$$(1.2) \quad a^{ij}(x) \partial_{ij} u(x) = f(x) \quad \text{in } \Omega$$

as well as the related partial differential equation in divergence form

$$(1.3) \quad -\partial_i (a^{ij}(x) \partial_j u(x)) = f(x) \quad \text{in } \Omega.$$

Received by the editors August 14, 2009.

2000 *Mathematics Subject Classification.* Primary 35B50, 35J15, 74B20.

Here and in the following, the summation convention of repeated indices is applied.

As Ω is bounded, we may assume throughout that there exists an open ball $B_d(0)$ with radius $d > 0$ around the origin such that

$$\Omega \subset B_d(0).$$

Weak maximum principle for classical solutions of (1.2).

Theorem 1.1. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be such that*

$$a^{ij}(x)\partial_{ij}u(x) \geq 0 \ (\leq 0) \quad \text{in } \Omega$$

and the coefficients matrix fulfils (1.1) with $\lambda \geq 0$. Then the maximum (minimum) of u in $\overline{\Omega}$ is achieved on $\partial\Omega$, that is

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad (\inf_{\Omega} u = \inf_{\partial\Omega} u).$$

Proof. See, e.g., [7, Section 3.1]. □

Theorem 1.1 can be put in divergence form provided that additionally $a^{ij}(x) \in C^1(\Omega)$, see [7, Section 3.6].

Weak maximum principle for strong subsolutions and viscosity solutions.

Let $u \in H^{2,p}(\Omega)$ solve

$$(1.4) \quad -a^{ij}(x)\partial_{ij}u(x) = f(x) \quad \text{in } \Omega.$$

A *strong subsolution* $u \in H^{2,p}(\Omega)$ is a function for which (1.4) holds with ' \geq ', i.e.

$$-a^{ij}(x)\partial_{ij}u(x) \geq f(x) \quad \text{in } \Omega.$$

For a function f , let $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\} = -\min\{f, 0\}$. We now recall the Alexandrov¹-Bakelman-Pucci maximum principle. For a proof in the case of strong solutions see [7], Section 9.1, and [3] in the case of viscosity solutions.

Theorem 1.2 (Alexandrov-Bakelman-Pucci maximum principle). *Let $f \in L^p(\Omega)$ and $u \in H^{2,p}(\Omega)$ be a strong subsolution of (1.4) and the coefficients matrix fulfils (1.1) with $\lambda > 0$. If $p \geq n$, there exists a constant C depending only on λ , Λ , p and n such that*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + Cd^{2-\frac{n}{p}} \left(\int_{\{\psi=\Gamma_{\psi}\} \cap \Omega} (f^+)^p dx \right)^{1/p},$$

where $\psi := u - \sup_{\partial\Omega} u^+$ and Γ_{ψ} is the concave envelope of ψ in $B_{2d}(0)$.

For $X \subset \mathbb{R}^n$ and a function $f : X \rightarrow \mathbb{R}$, the *concave hull* of f in X is the smallest concave function on the convexification of X minorised by f on X .

Remark 1.3. The following sufficient conditions guarantee the existence of a subsolution $u \in H^{2,2}(\Omega)$. If $\partial\Omega \in C^{1,1}$ and if a function $\varphi \in H^{1,2}(\Omega)$ exists such that $u - \varphi \in H_0^{2,2}(\Omega)$, this condition follows from an improved version of Theorem 8.12 in [7]. Conversely, if Ω is a convex domain, the $H^{2,2}(\Omega)$ -regularity of u follows from Theorem 2.3.2.9 in [8].

¹In the literature one also finds the transliteration Alexandroff of 'Александров'.

Theorem 1.2 is a classical result. Our focus lies on equations in the divergence form (1.3).

Weak maximum principle for weak subsolutions of (1.3).

A *weak subsolution* of (1.3) is a function $u \in H^{1,2}(\Omega)$ for which

$$(1.5) \quad \int_{\Omega} a^{ij}(x) \partial_j u(x) \partial_i v(x) \, dx \leq \int_{\Omega} f(x) v(x) \, dx$$

holds for all $v \in C_0^1(\Omega)$.

Remark 1.4. For the integrand on the left of (1.5) to be the weak formulation of the left hand side of (1.3), it is necessary that the divergence theorem holds for Ω , which we therefore assume tacitly in the following.

We state the following analogue of Theorem 1.2 for the problem (1.3). It is shown in [10].

Theorem 1.5 (Weak maximum principle for operators in divergence form). *Let (1.1) hold for $0 < \lambda \leq \Lambda$, let $f \in L^p(\Omega)$ with $p > \frac{n}{2}$ if $n \geq 4$ or $p = 2$ if $n = 1, 2, 3$, and let $u \in H^{1,2}(\Omega)$ be a weak subsolution of (1.3). Then there exists a constant C depending only on λ, n and p such that*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + Cd^{2-\frac{n}{p}} \left(\int_{\{\psi=Y_{\psi}\} \cap \Omega} (f^+)^p \, dx \right)^{1/p},$$

where $\psi := u - \sup_{\partial\Omega} u^+$ and

$$Y_{\psi} \in K_{\psi} := \left\{ w \in H_0^{1,2}(B_{2d}(0)) \mid w \geq \psi \text{ a.e. in } \Omega \right\}$$

is uniquely determined (see [10, Lemma 1.2]).

We mention a similar formulation of Theorem 1.5 with a different right hand side, i.e.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + Cr^{2-\frac{n}{p}} \|f\|_{L^p(\Omega)},$$

see [10, Theorem 3.1] and [11], [7] for a proof. Our argument in the proof of Theorem 2.1, showing the failure of Theorem 1.5 when locally $\lambda = 0$ applies for this altered formulation, too.

In contrast to Theorem 1.2, Y_{ψ} is not the concave envelope of ψ , but defined as the solution of the related obstacle problem

$$\text{Find } u \in K_{\psi} \text{ such that } J(u) = \inf_{w \in K_{\psi}} J(w),$$

where

$$J(w) := \int_{B_{2d}(0)} a^{ij} \partial_i w \partial_j w \, dx$$

for $w \in H_0^{1,2}(B_{2d}(0))$ and the set K_{ψ} defined by

$$K_{\psi} := \{w \in H_0^{1,2}(B_{2d}(0)) \mid w \geq \psi \text{ a.e. in } \Omega\}$$

with arbitrary $\psi \in H_0^{1,2}(\Omega)$, $\psi \leq 0$ on $\partial\Omega$.

As is shown in [10, Lemma 1.2], there exists a unique function $Y_\psi \in K_\psi$ with the property

$$J(Y_\psi) = \inf_{w \in K_\psi} J(w).$$

2. THE ASSUMPTION $\lambda > 0$ IN THEOREM 1.5 IS CRUCIAL

We shall prove by a counter example in space dimensions $n \leq 3$ that the assumption $\lambda > 0$ in Theorem 1.5 is essential and may not be replaced by $\lambda \geq 0$. We expect that our result holds for arbitrary space dimensions n . In general, Theorem 1.5 does not hold for elliptic operators which are not strictly elliptic. In particular, Theorem 1.5 is violated when the set

$$\{x \in \Omega \mid a^{ij}(x) = 0\}$$

has positive measure.

For this counter example we take resort to specific elastic energy functionals and make use of the explicit formulas for the relaxation of these functionals found in [4] for space dimensions $n = 2, 3$. For $n = 1$ we newly derive the explicit formula of the relaxed elastic energy in the appendix. Subsequently we introduce our example and illuminate the physical background.

The steady state of elastically stressed solids can be characterised as the minimiser of an elastic energy functional in the vector-valued variable $u \in H^{1,2}(\Omega; \mathbb{R}^n)$ that represents the deformation of the material with respect to the chosen reference state. Due to the inherent rotational symmetry (frame indifference), the Euler-Lagrange equation leads to an elliptic partial differential equation in the (linearised) strain $\varepsilon = \varepsilon(u) := \frac{1}{2}(\nabla u + \nabla^t u)$, which defines a symmetric $n \times n$ -matrix. Let $A:B := \text{tr}(A^t B)$ denote the scalar product between symmetric $n \times n$ matrices A, B . For the construction of the counter example we consider a solid consisting of two homogeneous constituents with different mechanical properties, as studied for instance in [1, 4]. For constituent $k \in \{1, 2\}$, the elastic energy density is given by

$$(2.1) \quad W_k(\tilde{\varepsilon}) := \frac{1}{2} \alpha_k (\tilde{\varepsilon} - \varepsilon_k^T) : (\tilde{\varepsilon} - \varepsilon_k^T) + w_k, \quad \tilde{\varepsilon} \in \mathbb{R}_{\text{sym}}^{n \times n},$$

and $\alpha_k, \varepsilon_k^T, w_k \geq 0$ are material parameters which denote the elasticity tensor, the transformation strain tensor and the value of the local minima, respectively.

The material can form microstructure which is described by a function $\tilde{d} : \Omega \rightarrow \{0, 1\}$ which gives the spatial distribution of the two phases. In consequence, the energy on the micro scale reads

$$(2.2) \quad W(\tilde{d}, \tilde{\varepsilon}) := \tilde{d} W_1(\tilde{\varepsilon}) + (1 - \tilde{d}) W_2(\tilde{\varepsilon}).$$

As introduced and explained in [4], the relaxation of the effective elastic energy functional for $d \in [0, 1]$ and $\varepsilon \in \mathbb{R}_{\text{sym}}^{n \times n}$ is given by

$$(2.3) \quad \widehat{W}(d, \varepsilon) := \inf_{\substack{\langle \tilde{d} \rangle = d \\ \tilde{d} \in \{0, 1\}}} \inf_{\tilde{u}|_{\partial\Omega} = \varepsilon x} \int_{\Omega} W(\tilde{d}, \varepsilon(\tilde{u})) \, dx,$$

where $\langle \tilde{d} \rangle := \int_{\Omega} \tilde{d}(x) \, dx := \frac{1}{|\Omega|} \int_{\Omega} \tilde{d}(x) \, dx$. The infimum over \tilde{d} is the result of homogenisation subject to the constraint that the volume fraction of the selected

phase is preset by d , see [6, Chapter 10]. This infimum is taken over functions \tilde{d} with bounded variation in Ω and values 0 or 1 a.e.,

$$(2.4) \quad \tilde{d} \in BV(\Omega; \{0, 1\}),$$

ensuring $\widehat{W} \geq 0$. Due to (2.4), the definition (2.3) is meaningful only for $d \in [0, 1]$. The second infimum in (2.3) is taken over functions $\tilde{u} \in H^{1,2}(\Omega; \mathbb{R}^n)$, where the condition $\tilde{u}|_{\partial\Omega} = \varepsilon x$ has to be read as $\tilde{u}(x) = \varepsilon x$ for a.e. $x \in \partial\Omega$.

Now we can show the main result which states that the assumption $\lambda > 0$ in Theorem 1.5 is essential and cannot be relaxed to $\lambda \geq 0$.

Theorem 2.1. *Let $n \leq 3$ and let (1.1) hold for $0 \leq \lambda \leq \Lambda$. If $u \in H^{1,2}(\Omega)$ is a weak subsolution of (1.3) for $f \in L^2(\Omega)$, the conclusion of Theorem 1.5 does not hold true.*

Proof. The minimisation problem (2.3) gives rise to an elliptic partial differential equation in divergence form (1.3) with coefficients in $L^\infty(\Omega)$, see Eqn. (2.5) below, where due to (2.4) the ellipticity condition (1.1) holds with $\lambda = 0$. We will show that the extension of Theorem 1.5 to $0 \leq \lambda$ contradicts the results in [4].

Using the above definitions, for given $d_0 \in [0, 1]$, $\varepsilon_0 \in \mathbb{R}_{\text{sym}}^{n \times n}$ and with the abbreviation $\tilde{d}_1 \equiv \tilde{d}$, $\tilde{d}_2 \equiv (1 - \tilde{d})$ it holds

$$\begin{aligned} \widehat{W}(d_0, \varepsilon_0) &:= \inf_{\substack{\langle \tilde{d} \rangle = d_0 \\ \tilde{d} \in \{0,1\}}} \inf_{\tilde{u}|_{\partial\Omega} = \varepsilon_0 x} \int_{\Omega} \tilde{d} W_1(\varepsilon(\tilde{u})) + (1 - \tilde{d}) W_2(\varepsilon(\tilde{u})) \, dx \\ &= \inf_{\substack{\langle \tilde{d} \rangle = d_0 \\ \tilde{d} \in \{0,1\}}} \inf_{\tilde{u}|_{\partial\Omega} = \varepsilon_0 x} \int_{\Omega} \sum_{k=1}^2 \tilde{d}_k \frac{\alpha_k}{2} (\varepsilon(\tilde{u}) - \varepsilon_k^T) : (\varepsilon(\tilde{u}) - \varepsilon_k^T) \, dx. \end{aligned}$$

We look at the minimisation problem in \tilde{u} , i.e. for fixed \tilde{d} we consider

$$E(\tilde{u}) := \int_{\Omega} \sum_{k=1}^2 \tilde{d}_k \frac{\alpha_k}{2} (\varepsilon(\tilde{u}) - \varepsilon_k^T) : (\varepsilon(\tilde{u}) - \varepsilon_k^T) \, dx$$

which we need to minimise over all $\tilde{u} \in H^{1,2}(\Omega; \mathbb{R}^n)$ with $\tilde{u}(x) = \varepsilon_0 x$, $x \in \partial\Omega$. The optimal u_{opt} is a solution of

$$(2.5) \quad \int_{\Omega} \sum_{k=1}^2 \tilde{d}_k \alpha_k (\varepsilon(u_{\text{opt}}) - \varepsilon_k^T) : \varepsilon(\zeta) \, dx = 0 \quad \text{for all } \zeta \in H_{\text{sym}}^{1,2}(\Omega; \mathbb{R}^n).$$

This yields the Euler-Lagrange equation

$$(2.6) \quad \begin{aligned} \operatorname{div} \left(\sum_{k=1}^2 \tilde{d}_k \alpha_k \varepsilon(u) \right) &= \sum_{k=1}^2 \alpha_k \varepsilon_k^T \nabla \tilde{d}_k, & \text{in } \Omega, \\ u(x) &= \varepsilon_0 x, & x \in \partial\Omega, \end{aligned}$$

which is a system of elliptic partial differential equations in divergence form with coefficients in $L^\infty(\Omega)$ that locally degenerate.

Next we want to apply Theorem 1.5 to this system of partial differential equations. To this end, we choose $\varepsilon_1^T = \varepsilon_2^T = 0$ and make the assumptions $(\nabla u)^t = \nabla u$

and $\alpha_k \in \mathbb{R}_{>0}$, $k = 1, 2$. This ensures that the system (2.6) decouples and that the right hand side is zero. Then (2.6) reads

$$(2.7) \quad \left. \begin{aligned} \operatorname{div} \left(\sum_{k=1}^2 \tilde{d}_k \alpha_k \nabla u_j \right) &= 0 && \text{in } \Omega, \\ u_j(x) &= (\varepsilon_0 x)_j, && x \in \partial\Omega, \end{aligned} \right\} 1 \leq j \leq n.$$

For each j , this corresponds to (1.3) with $f = 0$ and $a^{ij}(x) = \sum_{k=1}^2 \tilde{d}_k(x) \alpha_k$.

If Theorem 1.5 extended to $0 \leq \lambda$, it would hold $(u_{\text{opt}})_j(x) = (\varepsilon_0 x)_j$ in $\bar{\Omega}$ for a weak subsolution $(u_{\text{opt}})_j \in H^{1,2}(\Omega)$ of (2.7). Then also (2.5) would be fulfilled (as can be seen after partial integration) with $\varepsilon(u_{\text{opt}}) = \nabla u_{\text{opt}} = \varepsilon_0$ in $\bar{\Omega}$. So we would obtain

$$(2.8) \quad \begin{aligned} \widehat{W}(d_0, \varepsilon_0) &= \inf_{\substack{\langle \tilde{d} \rangle = d_0 \\ \tilde{d} \in \{0,1\}}} \int_{\Omega} \sum_{k=1}^2 \tilde{d}_k \frac{\alpha_k}{2} \varepsilon_0 : \varepsilon_0 \, dx \\ &= \sum_{k=1}^2 \frac{\alpha_k}{2} \varepsilon_0 : \varepsilon_0 \inf_{\substack{\langle \tilde{d} \rangle = d_0 \\ \tilde{d} \in \{0,1\}}} \int_{\Omega} \tilde{d}_k \, dx \\ &= d_0 W_1(\varepsilon_0) + (1 - d_0) W_2(\varepsilon_0). \end{aligned}$$

Identity (2.8) is in contradiction to the explicit formulas of $\widehat{W}(d_0, \varepsilon_0)$ for $n = 2, 3$ in [4] and for $n = 1$ derived in the appendix. For illustration, if $n = 2$, we have

$$(2.9) \quad \widehat{W}(d_0, \varepsilon_0) = d_0 W_1(\varepsilon_1^*) + (1 - d_0) W_2(\varepsilon_2^*) + \beta^* d_0 (1 - d_0) \det(\varepsilon_2^* - \varepsilon_1^*),$$

where $\beta^* \in \mathbb{R}$ and $\varepsilon_k^* \in \mathbb{R}_{\text{sym}}^{n \times n}$, defined in the appendix, depend in a complicated nonlinear way on d_0 , ε_0 and α_k , see the appendix for details. The formulas (2.8) and (2.9) coincide if and only if $\varepsilon_k^* = \varepsilon_0$, $k = 1, 2$. From $\varepsilon_1^* = \varepsilon_2^*$ we learn

$$(\alpha_2 - \alpha_1) \varepsilon_0 = (\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T) = 0$$

for all $\varepsilon_0 \in \mathbb{R}_{\text{sym}}^{n \times n}$, which yields $\alpha_1 = \alpha_2$. These conditions state that there is no dependence on the microstructure. Concluding, (2.8) and (2.9) are different in general.

Similarly, if $n = 1$, we obtain in the appendix equality of (2.8) (with d_0 replaced by d and ε_0 replaced by e) and (2.13) if and only if $e_k^* = e$, $k = 1, 2$, which yields the constraint $\alpha_1 = \alpha_2$. Hence, the two formulas for \widehat{W} are different in general. The same conclusion can be drawn for $n = 3$, taking resort to the explicit formula of $\widehat{W}(d_0, \varepsilon_0)$ in [4]. Hence the conclusion of Theorem 1.5 does not hold in general when $a^{ij}(x)$ degenerates. \square

APPENDIX

In this appendix we derive the explicit formula for $\widehat{W}(d, \varepsilon)$ if $n = 1$. First we recall the definitions of the regimes in 2D. Let the constant $\gamma^* > 0$ specify the range of feasible β , see [2], and let $\varphi(\beta, d, \varepsilon) := -\det(\varepsilon_2^* - \varepsilon_1^*)$. As explained thoroughly in [2], Regime 0, where there is no dependence on the microstructure, is characterised by $\varphi \equiv 0$; Regime I, where there are two optimal laminates of rank I, is defined by $\varphi(0, d, \varepsilon) > 0$; Regime III, where two optimal microstructures of rank II exist, is defined by $\varphi(\gamma^*, d, \varepsilon) < 0$; and finally Regime II, where there is a unique microstructure of rank I, is characterised by $\varphi(0, d, \varepsilon) \leq 0$ and $\varphi(\gamma^*, d, \varepsilon) \geq 0$.

We start from the explicit formula (2.9) in two dimensions and consider situations where the input data like applied macroscopic strain or the eigenstrains can be projected to a one-dimensional manifold. In the following we pick data constant in the y -direction such that $\varepsilon = \text{diag}(e, 0)$, $\varepsilon_1^T = \text{diag}(e_1^T, 0)$, $\varepsilon_2 = \text{diag}(e_2^T, 0)$ with $e, e_1^T, e_2^T \in \mathbb{R}$. Moreover, we require that the stress $\alpha_k \varepsilon$ is a diagonal matrix of the form $\text{diag}(\cdot, 0)$. This allows us to assume that $C_{1,12} = C_{2,12} = 0$, where we use a scaled version of the Voigt notation, see e.g. [1], and restrict ourselves to a cubic material.

Under these assumptions, we will check below that ε_1^* , ε_2^* are also contained in the selected one-dimensional submanifold of \mathbb{R}^2 .

Using this fact and the notations and definitions in [1] and recalling $\varphi(\beta, d, \varepsilon) = -\det(\varepsilon_2^* - \varepsilon_1^*)$, we first prove that Regime II is of no relevance in 1D. Indeed, in order to compute β_{II} for Regime II in 1D, we check that

$$\begin{aligned} (\varepsilon_1^* - \varepsilon_2^*)(d, \varepsilon) &= 0 \\ \Leftrightarrow (\alpha_2 - \beta Q)\varepsilon - (1-d)(\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T) &= (\alpha_1 - \beta Q)\varepsilon + d(\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T) \\ \Leftrightarrow (\alpha_2 - \alpha_1)\varepsilon &= (\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T). \end{aligned}$$

For $\alpha_1 = \alpha_2$ we are in Regime 0. Otherwise we obtain

$$(2.10) \quad \varepsilon = (\alpha_2 - \alpha_1)^{-1}(\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T).$$

By formula (2.10) only ε is determined, the other two parameters $\beta = \beta_{II}$ and d are still free. Thus Regime II does not occur. Furthermore, since rank-II-laminates cannot occur in 1D, Regime III does not play a role in this case. Hence β^* is identically zero in one dimension.

With $\beta^* = 0$, taking resort to the Voigt notation (see the appendix of [2]), we show that for $k = 1, 2$

$$\varepsilon_k^*(d, \varepsilon) = \text{diag}(e_k^*, 0)$$

with

$$(2.11) \quad e_1^*(d, e) = \frac{C_{2,11}e - (1-d)(C_{2,11}e_2^T - C_{1,11}e_1^T)}{dC_{1,11} + (1-d)C_{2,11}},$$

$$(2.12) \quad e_2^*(d, e) = \frac{C_{1,11}e + d(C_{2,11}e_2^T - C_{1,11}e_1^T)}{dC_{1,11} + (1-d)C_{2,11}}.$$

Indeed,

$$\begin{aligned} \alpha &= \text{diag}(dC_{1,11} + (1-d)C_{2,11}, dC_{1,11} + (1-d)C_{2,11}, 2dC_{1,44} + 2(1-d)C_{2,44}), \\ \alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T &= (C_{2,11}e_2^T - C_{1,11}e_1^T, 0, 0) \end{aligned}$$

and

$$\alpha_k \varepsilon = \begin{pmatrix} C_{k,11} & 0 & 0 \\ 0 & C_{k,11} & 0 \\ 0 & 0 & 2C_{k,44} \end{pmatrix} \begin{pmatrix} e \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_{k,11}e \\ 0 \\ 0 \end{pmatrix}.$$

Finally, use

$$\begin{aligned} \varepsilon_1^*(d, e) &= \alpha^{-1}[\alpha_2 \varepsilon - (1-d)(\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T)], \\ \varepsilon_2^*(d, e) &= \alpha^{-1}[\alpha_1 \varepsilon + d(\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T)]. \end{aligned}$$

From (2.9) we now obtain the formula in 1D

$$\begin{aligned}
(2.13) \quad \widehat{W}(d, e) &= dW_1(e_1^*) + (1-d)W_2(e_2^*) \\
&= d\frac{C_{1,11}}{2}\left(e_1^*(d, e) - e_1^T\right)^2 + (1-d)\frac{C_{2,11}}{2}\left(e_2^*(d, e) - e_2^T\right)^2 \\
&= d\frac{C_{1,11}}{2}\frac{\left((1-2d)C_{1,11}e_1^T + (e - (1-d)(e_1^T + e_2^T))C_{2,11}\right)^2}{(dC_{1,11} + (1-d)C_{2,11})^2} \\
&\quad + (1-d)\frac{C_{2,11}}{2}\frac{\left(C_{1,11}(e - d(e_1^T + e_2^T)) + (2d-1)C_{2,11}e_2^T\right)^2}{(dC_{1,11} + (1-d)C_{2,11})^2}.
\end{aligned}$$

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